Quality index for detecting reconstruction errors without knowing the signal in l$_0$-norm Compressed Sensing

C. A. SING-LONG[1,2], C. A. TEJOS[1,2], AND P. IRARRAZAVAL[1,2]

[1]DEPARTAMENTO DE INGENIERIA ELECTRICA, PONTIFICIA UNIVERSIDAD CATOLICA DE CHILE, SANTIAGO, R.M., CHILE, [2]BIOMEDICAL IMAGING CENTER, PONTIFICIA UNIVERSIDAD CATOLICA DE CHILE, SANTIAGO, R.M., CHILE

INTRODUCTION: Compressed Sensing (CS) ([1], [2], [3], [4]) is a recently created algorithm which allows reconstructing a signal from a small portion of its Fourier coefficients if that signal is sparse in a suitable basis. It was first used by Lustig et al. [5] in MRI, and it has become a popular alternative for speeding up the MRI acquisition processes. In practice, CS has been implemented as an l$_1$-norm minimization or a minimization of continuous approximations of an l$_0$-norm [6], [7]. The correct convergence of those minimization approaches is guaranteed only when the number of acquired samples in the Fourier domain is bigger than a certain quantity that depends on the size of the support of the original signal (the number of non-zero coefficients in the sparse domain of the signal). This constitutes a major problem as the support of the signal is not known, and therefore, there is no information to judge whether the reconstructed signal is correct or not. In this article we present a mathematical index that can discriminate correct from erroneous CS reconstructions (without knowing the original signal), and therefore identify the presence of reconstruction artifacts.

METHODS: Following the approach presented in [6] and [7] we solved a CS problem by minimizing iteratively continuous approximations of the l$_0$-norm. We approximated it with $\rho_\sigma(|x|) = 1 - 1/(|x|/\sigma + 1)$ and we used a different fixed-point minimization as that suggested in [6]. We undersampled the data in the phase-encoding direction (rows) and we reconstructed column by column. We call f a column to be reconstructed in the image domain and M the number of acquired samples from a total of N Fourier coefficients. Let c be the vector with $N - M$ unknown Fourier coefficients and v the minimum l$_2$-norm approximation of f for the given Fourier coefficients. We define $u(c) = v + Pc$ as the approximation to f, where P is the $N \times (N - M)$ matrix whose columns are the Fourier basis associated with the unknown coefficients. Finally, let $D(c)$ be the $N \times N$ diagonal matrix whose entries are related to the derivative of ρ_σ, evaluated on the components of $u(c)$. The first order optimality condition (FOOC) is:

$$P^* D(c) Pc = -P^* D(c) v$$

We see that the FOOC constitutes a fixed-point equation. Also, the matrix $P^* D(c) P$ is positive-definite. Let λ^- be the minimum eigenvalue of $P^* D(c) P$ and $\bar{\lambda}$ the mean value of its eigenvalues. This fixed-point minimization allows us to evaluate the following index:

$$\eta = 1 - \lambda^-/\bar{\lambda}$$

As will be demonstrated, this index shows a high correlation with the reconstruction error, measured as $\sqrt{\sum_n |f_n - g_n|^2} / \sqrt{\sum_n |f_n|^2}$, with g being the reconstructed column. To evaluate the effectiveness of our index, we generated 10 vectors of size $N = 96$, with integer intensities randomly chosen from a uniform distribution $\{0 - 255\}$. With each vector we constructed 24 sparse signals with different support size S by choosing randomly $1, 5, \ldots, 89, 93$ data points and changing the rest to zero. Sparse signals were Fourier transformed and undersampled at random locations with 24 different undersampling rates (the number of sampled coefficients were $M = 2, 6, \ldots, 90, 94$), so that the total reconstruction experiments were $10 \times 24 \times 24 = 5760$. At each final iteration step we calculated the index η. We also performed a second experiment reconstructing a 96×96 – pixel MR image of a head, quantized in 8-bits with $95\%, 90\%, 80\%, \ldots, 10\%$ and 5% of the readout rows. For each undersampling rate 20 reconstructions were made, each time with sampled coefficients chosen at uniform random locations, leading to a total of $96 \times 11 \times 20 = 21120$ column reconstructions. For this experience we used a vertical finite differences sparsity transform T for each column. In this case the matrix $D(c)$ is related to the components of $Tu(c)$. For both experiments, we calculated the reconstruction error and the index η. We also considered a reconstruction exact if the error was less than 10^{-2} and a threshold $\eta = 0.95$ to establish *a priori* if the reconstruction was exact. We performed a study of the false positives ($\eta > 0.95$ for an error inferior to 10^{-2}) and false negatives ($\eta \leq 0.95$ for an error superior to 10^{-2}) obtained. It is also important to notice that the index η reaches a value very close to the final value in only a few iterations.

FIGURE 1

FIGURE 2

TABLE 1	RANDOM VECTOR RECONSTRUCTION	MR IMAGE RECONSTRUCTION
GOOD DETECTIONS (%)	5534 (96%)	20542 (97.3%)
FALSE POSITIVES (%)	205 (3.6%)	326 (1.5%)
FALSE NEGATIVES (%)	21 (0.4%)	252 (1.2%)

RESULTS: As can be seen in Fig.2 for $\eta > 0.95$ the error increases sharply, whereas for $\eta \leq 0.95$ the error stays well under 0.2. Table 1 shows that choosing the threshold $\eta > 0.95$ produces a 96% of correct detections, while maintaining a small percentage of false positives and false negatives for both experiments. For the MR image, the percentage of false positives halved and the number of false negatives doubled. Let us notice that, in average, this index produces more false positives than false negatives, so the estimations are rather pessimistic. Fig.1 shows how the index η may be used to detect columns where the reconstruction is not exact and, in consequence, where visual artifacts may be present.

CONCLUSIONS: This work introduces a modified version of the fixed-point algorithm in [6]. It also introduces an index that shows a high correlation with the reconstruction error. We emphasize that this index does not require any knowledge *a priori* of the original signal, thus allowing detecting reconstruction errors and visual artifacts due to undersampling. Our experiences also show that this can be extended to a finite differences sparsity basis. In future work we will study how these ideas may be extended to other sparsity basis, such as wavelets, and what are the underlying structures that produce false positives or false negatives.

[1] CANDÈS ET AL, IEEE TIT, 52(2), PP. 489 − 509, [2] CANDÈS ET AL, IEEE TIT, 51(12), PP.4203 − 4215, [3] DONOHO, IEEE TIT, 52(4), PP.1289 − 1306, [4] CANDÈS ET AL, INV. PROB., 23(3), PP.969 − 985, [5] LUSTIG ET AL, MRM, 58(6), PP.1182 − 1195, [6] TRZASKO ET AL, IEEE TMI, *IN PRESS*,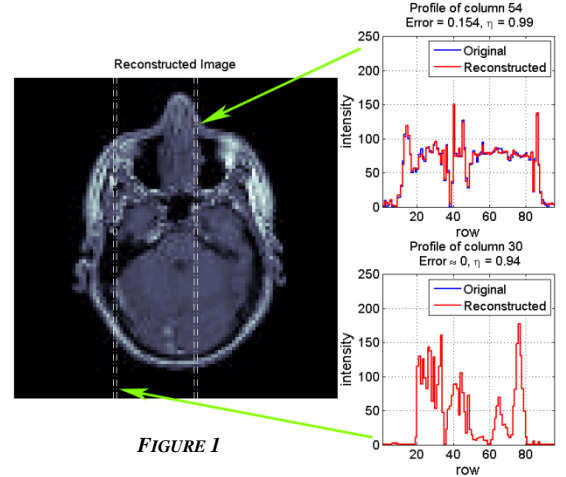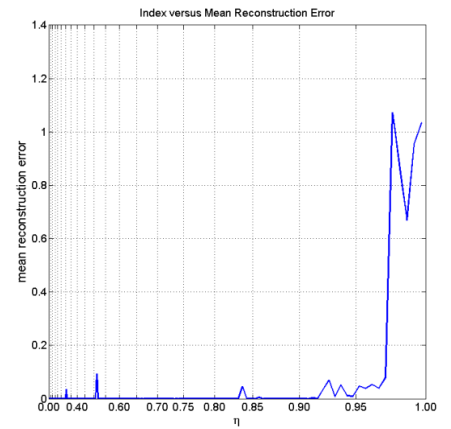 [7] MOHIMANI ET AL, IEEE TSP, *IN PRESS*.