Matrix Formulation and Tikhonov Regularization of HYPR Reconstruction

M. Haeberlin[1,2], S. Skare[1], R. D. Newbould[1], K. P. Pruessmann[2], and R. Bammer[1]

[1]Department of Radiology, Stanford University, Stanford, CA, United States, [2]Institute for Biomedical Engineering, ETH Zurich, Zurich, Switzerland

Introduction: HYPR (HighlY constrained backPRojection) has previously been proposed by Mistretta *et al.* [1] as a non-iterative image reconstruction method for time-resolved MRA and has evolved ever since to tackle a multitude of reconstruction problems including less sparse data, such as those obtained with perfusion or diffusion imaging [7, 6]. The HYPR algorithm, however, exhibits an inherent inconsistency between the acquired projection data and the HYPR-synthesized projection data. To minimize this inconsistency Griswold *et al.* [2] recently proposed a Conjugate Gradients (CG) approach. Although self-regularizing, the CG method exhibits inherent instability in ill-posed problems which requires further regularization methods. A simple approach to stabilize the solution is Tikhonov regularization [8]. Tikhonov regularization has previously been proposed not only for MRI but also for backprojection-data, such as obtained with PET and CT [3, 4]. In extension to the work of Griswold [2] and Older [5], we put the backprojection reconstruction into a matrix formalism framework. The matrix formalism helps better understand HYPR and its variant, CG-HYPR. Moreover, it allows one to incorporate regularization techniques as demonstrated in this work by including Tikhonov regularization.

Material and Methods: Note that the HYPR reconstruction is linear and can thus be expressed as a sequence of matrix operations as follows:

$$\mathbf{HYPR = C * R^H * Pc}, \qquad [1]$$

Pc is $\mathrm{diag}\{1/Pc_{(i,k),(i,k)}\}$ with the elements $Pc_{(i,k),(i,k)}$ being the k^{th} element of the i^{th} projection of the composite image. Furthermore, $\mathbf{R^H}$ is the backprojection operator and corresponds to the Hermitian of the Radon Transform matrix, and **C** is the composite operator and is expressed as $\mathrm{diag}\{Composite_{j,j}\}$ with $Composite_{jj}$ being the j^{th} element of the composite image. This formalism is illustrated in Fig. 1.

The Hermitian of the HYPR-operator is then given by:

$$\mathbf{HYPR^H = Pc^H * R * C^H}. \qquad [2]$$

As proposed in [2], CG-HYPR minimizes data inconsistencies seen in regular HYPR by minimizing the norm $\|\mathbf{R*HYPR(Pi) - Pi}\|^2$, where **R** denotes the Radon Transform applied to the HYPR image. The augmented CG algorithm in matrix form is illustrated in Fig. 2.

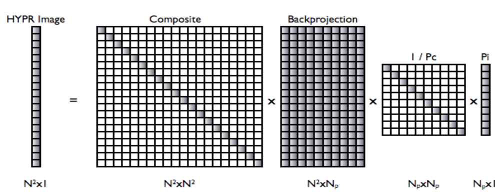

Figure 1 - Matrix representation of HYPR. *P_i represents the acquired data for the current timeframe, backprojection is the transpose of the Radon Transform, and pixel-by-pixel multiplications are represented as diagonal matrices. N_p denotes the total number of projection samples, N^2 is the number of pixels in the HYPR image.*

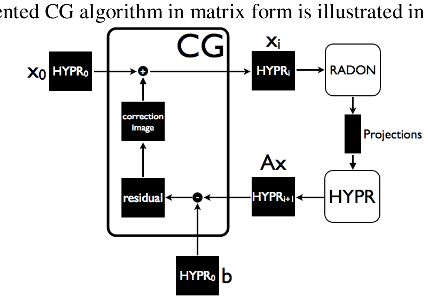

Figure 2 - Block diagram of the CG method used to minimize data inconsistency in HYPR. *The CG algorithm iteratively updates our initial guess x_0 by computing a correction image from the residual vector.*

With the presented framework it is possible to incorporate any kind of regularization technique into the CG-HYPR algorithm. For example, introducing the norm of the reconstructed projections, $\|\mathbf{x}\|^2$, as an additional penalty term, the minimization problem can be recast as follows:

$$\min \{ \|\mathbf{R * HYPR(x_i) - Pi}\|^2 + \alpha\|\mathbf{Bx_i}\|^2 \}.$$

B is an arbitrary matrix and α is an arbitrary, positive scalar that controls for the amount of regularization. The setting of α was done manually but if needed can be done in an automated yet more computationally intense mechanism by L-curve or crossvalidation. Ultimately, solving for **x** yields

$$\mathbf{x_s = (HYPR^H * R^H * R * HYPR(x_i) + \alpha B^H * B)^{-1} * HYPR^H * R^H * P_i} \qquad [3]$$

which can be done with the CG method.

Simulations have been performed in Matlab using a software phantom with simple structure whose intensity varies over time and basically resembles a first pass plus recirculation of contrast agent within arteries and (with time delay) veins. Parameters were chosen as follows: $\alpha = 0.001$, $\mathbf{B} =$ Identity, $N = 256$, 35 timepoints, 30 projections per timepoint and 10 CG iteration steps for each timepoint.

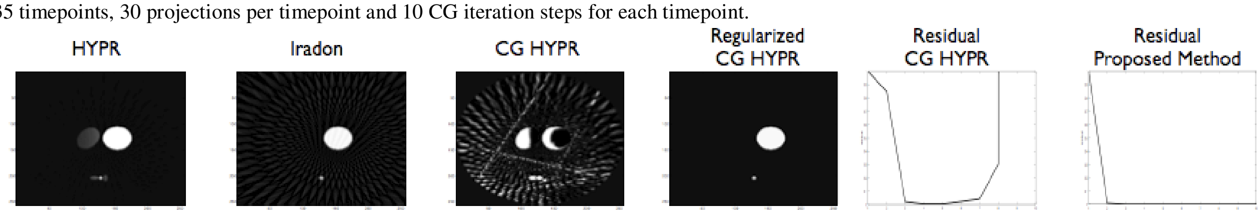

Figure 3 – A comparison between HYPR, common filtered backprojection (Iradon), CG-HYPR and the proposed method (Regularized CG HYPR). *The main advantage of the proposed method is in the convergence behavior of the CG, which is stable. It also inherits HYPR's ability to suppress streak artifacts common in inverse Radon reconstructions. Without regularization, CG diverges and yields unacceptable results.*

Results: The proposed method stabilizes CG-HYPR and exhibits faster convergence (3-6 iterations). Tikhonov Regularization makes CG-HYPR practicable due to its stable residual behavior. As shown in Fig. 3, CG-HYPR yields unacceptable results if not stopped in one of the few converged states. Like CG-HYPR, the proposed method eliminates streak artifacts seen in filtered backprojection reconstruction and enhances temporal resolution.

Discussion: The proposed matrix formalism allows incorporating regularization techniques into CG-HYPR. A key issue is finding the correct regularization parameter α which requires further investigation. Regularization techniques may also drive down the minimum of the residual function, thus increasing image quality.

References: [1] CA Mistretta *et al.*, MRM, 55, 2006, p. 30 [2] MA Griswold *et al.*, Procs. ISMRM 2006, p. 188. [3] L Kaufmann, A Neumaier, IEEE Trans. Med. Im, 3, 1996, p. 385. [4] EL Piccolomini, F Zama, Appl. Math. Comput, 102, p. 87. [5] JK Older, PC Johns, Phys. Med. Biol, 8, 1993, p. 1051. [6] AL Alexander *et al.*, Procs. ISMRM 2006, p. 192. [7] O Unal *et al.*, Procs. ISMRM 2007, p. 495. [8] AN Tikhonov, Sov. Math.-Dokl. 4, 1963, p. 1035.

Acknowledgments: This work was supported in part by the NIH (2R01EB002711, 1R21EB006860, P41RR09784), the Lucas foundation, the Oak foundation, and the Swiss-American Society.