Homotopic L_0-Minimization for Highly-Undersampled MRI Reconstruction

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Introduction Compressed or Compressive Sensing (CS) [1,2,3] and related L_1-minimization techniques [4,5,6] can accurately and efficiently reconstruct sparse or compressible magnetic resonance images (MRI) even at sampling rates far below the Nyquist limit. Although L_1-based formulations of the partial k-space recovery problem have the advantage of convexity and thus a readily-achievable global minima, they inherently require the image to be modestly oversampled above the theoretical minimum sampling rate to assure their efficacy [7]. In this work, we introduce an alternative reconstruction paradigm based on homotopic L_0 semi-norm minimization which asymptotically diminishes this oversampling factor and allows for MR images to be reconstructed from even fewer samples than are necessary for L_1-based methods.

Theory Let f be an MR image of interest and define Φ as a Fourier undersampling operator (e.g. Cartesian random phase encoder [8]). Additionally, let Ψ represent a sparsifying transformation such as a wavelet or differential operator. Ideally, a signal possessing a K-sparse representation can be recovered from as few as $2K$ random samples in some incoherent domain by solving the so-called L_0-minimization problem,

$$\tilde{u} = \arg\min_u \sum_\Omega 1\left(|\Psi u| > 0\right) \text{s.t. } \|\Phi u - \Phi f\|_2^2 \leq \varepsilon^2, \quad (1)$$

where $\mathbf{1}(|\cdot|>0)$ denotes the indicator function. Unfortunately, this problem is NP-complete and, as $\mathbf{1}$ possesses a zero gradient almost everywhere (a.e.), even local minima cannot be readily computed. The L_1-minimization problem is obtained by relaxing $\mathbf{1}$ to the modulus function, and this formulation generally necessitates acquisition of at least $3K$-$5K$ measurements to ensure that an accurate reconstruction is achievable [7]. Alternatively, consider

$$\tilde{u} = \arg\min_u \lim_{\sigma\to 0} \sum_\Omega \rho\left(|\Psi u|, \sigma\right) \text{s.t. } \|\Phi u - \Phi f\|_2^2 \leq \varepsilon^2, \quad (2)$$

where ρ denotes a dynamic concave semimetric (e.g. the Laplace error function or L_p semi-norm class for $0<p<1$) that is homotopic with, or can be continuously deformed into, $\mathbf{1}$ as $\sigma\to 0$. Correspondingly, as σ is diminished and ρ provides an increasingly better approximation of $\mathbf{1}$, the oversampling factor above the $2K$ limit is also progressively reduced [9,10]. While similar mathematical approaches have been applied to the shape-from-shading problem in computer vision [11] and for Bayesian tomography [12], the authors believe this to be both the first application of homotopic approximation to the CS problem as well as to MRI reconstruction.

Methods Although (2) is highly non-convex and thus exhibits no guarantee of a practically-achievable global minima, ρ, unlike $\mathbf{1}$, possesses a non-zero gradient a.e. and thus standard descent methods readily admit local minima. Furthermore, combination of a robust numerical solver such as a half-quadratic (HQ) or fixed-point iteration (FPI) [10,13] in conjunction with a standard continuation scheme for reduction of σ [14] generally yields local minima which are visually superior to the global results obtained by L_1-minimization. Note that both separate real and imaginary [5] or magnitude and phase regularizations [15] are possible.

Example An example juxtaposition of L_1 and homotopic L_0-minimization for sparse MRI reconstruction is shown in Figure 1. The shown MRI phantom was retrospectively undersampled by 78% via a Cartesian random phase encode mask [8] and both solutions were computed using the aforementioned FPI numerical scheme with a gradient sparsity operator. In both cases, < 30 outer iterations of the FPI scheme were required yielding a Matlab© reconstruction time of ~2 min on a standard desktop PC (3GHz Pentium IV w/ 1GB memory) for the 256x256 example which is comparable to the results presented in [3-6,8]. While both L_1 and homotopic L_0-minimization provide a marked improvement over the zero-filled reconstruction, note the superiority of the proposed method in terms of both overall contrast and intensity uniformity as well morphological detail as highlighted in the enlargements images. Comparable results have been obtained for other image types as well as with alternative sampling patterns.

Discussion In this work, we have proposed a novel technique for sparse MRI reconstruction from highly-undersampled k-space data that allows for accurate reconstructions to be achieved at sampling rates even lower than are required by L_1-based techniques. For specialized imaging applications such as time-resolved [16] and parallel imaging [6] where the superiority of L_1-based methods over conventional linear techniques has already been demonstrated, homotopic L_0-minimization may allow for even greater reduction of scan time.

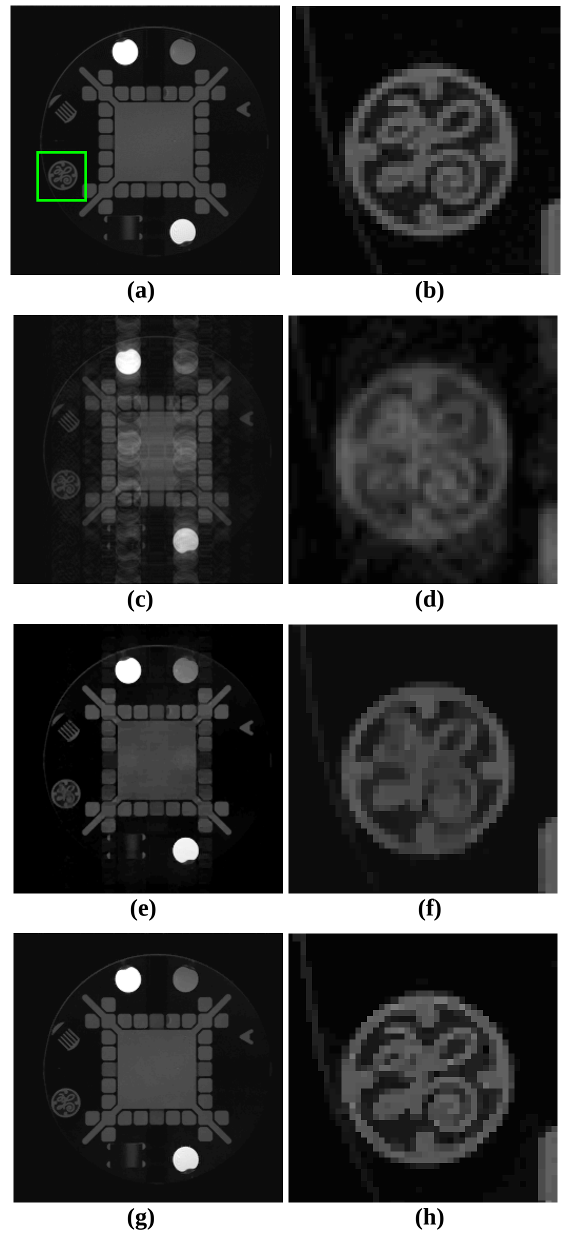

Figure 1. *Example partial k-space reconstructions. (a-b) show a fully-sampled phantom image and an enlargement of the region highlighted in green. (c-h) show images that have been undersampled by 78% (random phase encoding) and reconstructed by (c-d) zero-filling, (e-f) L_1-minimization, and (g-h) homotopic L_0-minimization.*

References

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