Generalized Two-Dimensional Orthogonal Spatial Encoding Fields

G. Schultz[1], A. Welz[1], J. Hennig[1], and M. Zaitsev[1]
[1]Dept. of Diagnostic Radiology - Medical Physics, University Hospital Freiburg, Freiburg, BW, Germany

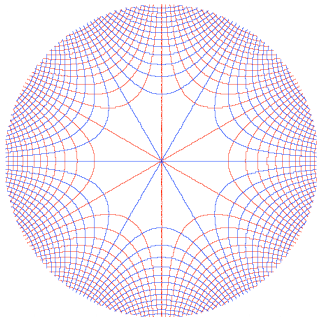

Figure 1: Superposition of two contour plots of spatial encoding fields with six poles. The fields vary sinusoidally in the circumferential direction. They are phase-shifted by 90°. This results in contour lines – or equivalently in field gradients – which are mutually orthogonal in each point.

Introduction: The PatLoc concept has been introduced to perform imaging using non-bijective, curvilinear encoding fields [1]. Simulations have shown that a pair of mutually rotated multipolar encoding fields with cylindrical symmetry combined with a linear gradient for slice-selection have very good imaging properties especially in the outer half of the enclosed cylinder [2]. The gradients of these fields are mutually orthogonal as depicted in Fig. 1. Not least, this property is responsible for the good quality of the reconstructed images.
We have addressed the problem of finding a simple method capable of generating all possible orthogonal fields for 2D encoding. Basic findings from complex analysis show that it is possible to describe the fields as real and imaginary parts of a complex-valued holomorphic function. Moreover, the set of holomorphic functions on a given simply connected region is equivalent to all mutually orthogonal source free fields on the same region.

Theory: A holomorphic function f(z) can be split up into its real and imaginary part and at the same time the real and imaginary compent of z = x+iy can be interpreted as spatial components in two-dimensional Euclidian space:

$$f(z) = u(x,y) + iv(x,y) \qquad (1)$$

If f is holomorphic, u and v satisfy the Cauchy-Riemann differential equation. It can also easily be shown that u and v satisfy Laplace's equation and at the same time $(\nabla u)(\nabla v) = 0$. On the other hand an arbitrary function satisfying Laplace's equation on a simply connected region can be written as the real or imaginary part of a holomorphic function [3]. The correspondance of encoding fields B_1 and B_2 with u and v

$$B_1(x,y) = \mathrm{Re}(f(z)) = u(x,y) \quad \text{and} \quad B_2(x,y) = \mathrm{Im}(f(z)) = v(x,y) \qquad (2)$$

leads to the following interpretation: every mutually orthogonal pair of magnetic encoding fields can be described by a corresponding holomorphic function and vice versa.

Results: Some simple applications to the above theoretical approach are given in this section. Within a neighbourhood of the origin, every holomorphic function coincides with its Taylor series $f(z) = \sum a_n z^n$. It is especially interesting to examine the fields, which correspond to the monomials z^n, $n = 1,...,\infty$. For $n = 1$ this gives linear gradients, as $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$. For $n > 1$ the monomials are given by $z^n = r^n e^{in\varphi}$ and therefore $B_1^n(r,\varphi) = r^n \cos(n\varphi)$, $B_2^n(r,\varphi) = r^n \sin(n\varphi)$. Hennig et al. has proposed in (2) to use coils, which correspond to the monomial of order $n = 4$. Fig. 2 illustrates the linear gradients and two pairs of ideal multipolar fields corresponding to the monomials of lowest order. These fields were used to encode a phantom of concentric circles and radial lines. The simulated signals served as input to a generalized SENSE-like reconstruction method, which is capable of dealing with the occurring ambiguities. The reconstructed images are also presented in Fig. 2.

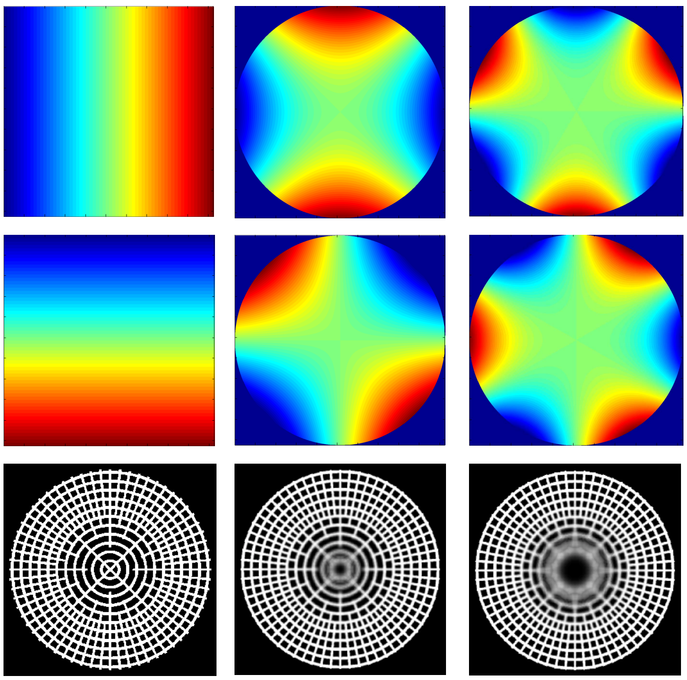

Figure 2: Upper line and middle line: The orthogonal magnetic fields corresponding to the monomials of lowest order. Bottom line: Reconstructed images of a phantom with concentric circles and radial lines having been encoded by the encoding fields above.

Discussion: We have shown that it is useful to represent all possible orthogonal fields with a cylindrical symmetry by holomorphic functions. Only encoding fields corresponding to monomials have so far been used or proposed for imaging. Expansion into a Taylor series points these fields out**.** They are closely related to spherical harmonics expansions used in gradient and shim coil design [4]. The fields corresponding to the monomials z^n are in fact the fields of shim coils of order (n,n).
There is a close relationship between radial decay of field strength and multipolarity. When sharp imaging is only desired at the periphery, a higher degree of multipolarity leads to higher resolution. On the other hand, when large imaging areas are important, the field should change more or less linearly. In this case non-bijective imaging could possibly be improved by relaxing the condition of strict orthogonality. Moreover, in practical situations the boundary conditions to Laplace's equation can in general not be met exactly because coils have to be open at least on one side.
The presented approach suggests that there exists a broad spectrum of possible magnetic fields, which still have to be investigated. To give an example, the function $f(z) = \sin(z)$ corresponds to fields, which vary sinusoidally, but not in the cirumferential direction, but in a linear fashion.
On the one hand, the flexibility of the presented approach is well adapted to the optimal design of encoding fields for specialized applications, on the other hand, the simple mathematical structure makes this approach especially interesting for embedding into more general numerical optimal design formulations.

Acknowledgements: This work is part of the INUMAC project supported by the German Federal Ministry of Education and Research, grant #13N9208.

References: [1] Hennig et al., ISMRM, 2007, 453; [2] Hennig et al., submitted to MAGMA; [3] Boas, Mathematical Methods in the Physical Sciences, 1983; [4] Turner, Magn Reson Imaging, 1993, 903-920.