Three-Dimensional Isotropic Filter Design with Arbitrary Pass-Band and Stop-Band Specifications

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Introduction: Variable-density sampling of k-space with accompanying proportional filtering, offers improved spatial impulse response without SNR tradeoffs as demonstrated for imaging [1] and spectroscopic imaging [2]. Prior work [2] relied on 1-D filter specifications, a widely investigated topic, while the general three-dimensional case is much less studied. While three dimensional filters can be derived by simple transformations from 1D, optimality of the filter will in general be lost. We present here an algorithm for the direct design of an optimal, spherically symmetric, three-dimensional filter with arbitrary extent in k-domain, and pass-band (i.e. voxel size) and stop-band (side lobe suppression) specified in the image domain. While the methods presented are general, a particular motivation for this work is the suppression of in-brain lipid signals from adjacent, undesirable, and strong subcutaneous lipids by radially-symmetric spatial impulse response.

$F_p(m,i) = \sum_{x^2+y^2+z^2=r_p(i)^2} \exp(-j2\pi(k_{px}(m)x + k_{py}(m)y + k_{pz}(m)z)/N)$	(1)
$F_s(m,i) = \sum_{x^2+y^2+z^2=r_s(i)^2} \exp(-j2\pi(k_{sx}(m)x + k_{sy}(m)y + k_{sz}(m)z)/N)$	(2)
$f = \arg\min_f \frac{f^H F_s^H F_s f}{f^H F_p^H F_p f}$	(3)
$(F_s^H F_s)(k,i) = \sum_{\substack{x_k^2+y_k^2+z_k^2=r_s(k)^2 \\ x_i^2+y_i^2+z_i^2=r_s(i)^2}} \sum_m \exp(-j2\pi(k_{sx}(m)(-x_k+x_i) + \ldots$ $k_{sy}(m)(-y_k+y_i) + k_{sz}(m)(-z_k+z_i))/N)$	(4)
$(F_p^H F_p)(k,i) = \sum_{\substack{x_k^2+y_k^2+z_k^2=r_p(k)^2 \\ x_i^2+y_i^2+z_i^2=r_p(i)^2}} \sum_m \exp(-j2\pi(k_{px}(m)(-x_k+x_i) + \ldots$ $k_{py}(m)(-y_k+y_i) + k_{pz}(m)(-z_k+z_i))/N)$	(5)
$\frac{f^H F_s^H F_s f}{f^H F_p^H F_p f} = \frac{g_1^H A g_1 + g_1^H B g_2 + g_2^H C g_2}{g_1^H D_1 g_1}$	(6)
$\frac{g_1^H A g_1 + g_1^H B g_2 + g_2^H C g_2}{g_1^H D_1 g_1} = \frac{g_1^H S g_1}{g_1^H D_1 g_1}$	(7)
$\frac{g_1^H S g_1}{g_1^H D_1 g_1} = \frac{g_3^H D_2^{-1} S D_2 g_3}{g_3^H g_3}$	(8)

Methods:
Our filter design method is based on an optimization algorithm that minimizes the ratio of the energy in filter stop-band to the energy in the pass-band as outline in equation (3). The design method requires a specification of the k-space extent, and the setting of pass-band and stop-band in the image.

The extent of the filter in k-space determines the number of nonzero filter coefficients. Due to the radial symmetry, the filter has the same coefficients at a fixed radius. The response of each independent parameter is determined by the DFT of the signal obtained by setting that parameter to one and zero elsewhere, i.e. one at a certain radius and zero elsewhere. From the responses, we can represent, in a matrix form, the pass-band and stop-band responses as a multiplication of a matrix whose column is the response of each parameter and a vector of coefficients. The coefficients of the matrices can be represented as equations (1, 2).

However, computing all the coefficients of the two matrices is impractical since it requires one three-dimensional FFT to compute each coefficient. The criterion of minimum energy ratios is achieved by the filter coefficients in equation (3). As we can see from this expression, we need to calculate not the matrix itself, but rather the inner product between any two columns of the matrix. By changing the order of summation (4, 5), we can reduce the amount of computation. This computation requires only one three-dimensional FFT, a dramatic savings. We can also reduce the amount of computation by taking advantage of the symmetry along the cardinal axes, x, y, and z. After calculating the inner products, by performing singular value decomposition (SVD), we can further simplify the minimization process (6) for some matrices A , B , and C .

We can minimize this simplified form (6) in two steps. Assuming g_1 is fixed, since the numerator is the second order polynomial of g_2, we can find a g_2 that minimizes the numerator. In fact, for any g_1, g_2 will be $g_2=Q\, g_1$ for a certain fixed matrix Q. After substituting g_2, the minimization can be represented as (7) for some matrix S. Since the singular values are non-negative, the diagonal entries of D_1 are all positive. Thus, there exists a diagonal matrix D_2 in the notation above. Then, the minimization can be represented as (8). This is minimized when g_2 is the singular vector of $D_2^{-1} S\, D_2$ corresponding to the minimum singular value.

Notations:
F_p, F_s : the matrices for pass-band and stop-band.
$k_{px}(m)$, $k_{py}(m)$, $k_{pz}(m)$: the m[th] k-space point in the pass-band
$k_{sx}(m)$, $k_{sy}(m)$, $k_{sz}(m)$: the m[th] k-space point in the stop-band
$r_p(i)$: the i [th] smallest radius in the pass-band
$r_s(i)$: the i [th] smallest radius in the stop-band
f : the vector of the optimal filter coefficients
The SVD of $F_p^H F_p : F_p^H F_p = V^H D V$
D_1 : a diagonal matrix whose entries are positive singular values of $F_p^H F_p$
D_2 : a diagonal matrix satisfying $D_2^H D_2 = D_1$
$g : g = Vf$
g_1 : the first $rank(D)$ entries of g
g_2 : the last $\dim(D) - rank(D)$ entries of g
$g_3 : g_3 = D_2 g_1$

Results: For the specific goal of subcutaneous lipid suppression in brain spectroscopic imaging, we set as pass-band only the discrete DC point. With FOV of 24 cm, the pairs of the voxel size and the 40-dB decay-distance of the filters are the followings: (1.06cc, 1.13cm), (0.73cc, 1.06cm), (0.5cc, 0.84cm), (0.36cc, 0.80cm), and (0.15cc, 0.53cm). These results were achieved in computation time within a few minutes on a standard PC with C++.

Conclusion: We have derived an optimal three dimensional filter design based on an energy ratio between stop- and pass-bands. The full optimization was reduced substantially in computation cost to achieve less than a one minute optimization time on a standard PC.

References: [1] Parker et al, Medical Physics, 14(4), 640-645, 1987. [2] Adalsteinsson et al, MRM 42:314-323, 1999.