SETS: Simultaneous Equations with Taylor Expansions in Undersampled Cartesian Data

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Introduction

Imaging time constraints often prevent the acquisition of a full k-t space for dynamic objects. In many instances, only part of the k-t space is acquired and the missing information can be approximated using various techniques. UNFOLD is a method which assumes that more than one spatial point can share the same temporal bandwidth without overlap (1). This work presents a method based on the theory of UNFOLD that allows some degree of overlap in the temporal bandwidth. This can be accomplished with a Taylor series approximation of the signal intensities from each spatial point in the image.

Theory

It is most intuitive to consider a simple 1D example and then extend the results into a more generalized form. Start with a 1xN image with $m_n(t)$ defined as the complex intensity of pixel n, at time frame t, and $M_k(t)$ as the Fourier Transform at each time frame. The k-t space is then undersampled by only acquiring every N^{th} k-space line, with the sampling pattern being shifted one line each time frame (Figure 1). The resulting aliased signal intensities at each pixel are given by

$$\tilde{m}_n(t) = \frac{1}{N}\sum_{j=0}^{N-1} m_j(t)\exp\left[\frac{-2\pi i}{N} t(j-n)\right] \quad [1]$$

A sliding window image can then be formed as

$$p_n(t) = \sum_{f=0}^{N-1}\tilde{m}_n(t+f) = \frac{1}{N}\sum_{f=0}^{N-1}\sum_{j=0}^{N-1} m_j(t+f)\exp\left[\frac{-2\pi i}{N}(t+f)(j-n)\right] \quad [2]$$

Utilizing the Fourier Transform in time

$$P_n(k) = \sum_{t=0}^{T-1} p_n(t)\exp\left[\frac{-2\pi i}{T}kt\right] = \sum_{j=0}^{N-1} M_j\left(k+\frac{T(j-n)}{N}\right)A_{nj}\left(k+\frac{T(j-n)}{N}\right) \quad [3]$$

Where

$$M_j(k) = FFT\{m_j(t)\} \quad A_{nj}(k) = \frac{1}{N}\sum_{f=0}^{N-1}\exp\left[\frac{-2\pi i}{N}f(j-n)\right]\exp\left[\frac{-2\pi i}{T}kf\right] \quad [4]$$

Then let

$$\vec{P} = P_n\left(k+\frac{Tn}{N}\right), \quad \vec{M} = M_j\left(k+\frac{Tj}{N}\right) \quad and \quad \ddot{A} = A_{nj}\left(k+\frac{Tj}{N}\right) \quad [5]$$

A solution for $M_k(t)$, and therefore $m_n(t)$, can are found as the solution to a set of simultaneous linear equations

$$\vec{P} = \ddot{A}\cdot\vec{M} \quad or \quad \vec{M} = \ddot{A}^{-1}\cdot\vec{P} \quad [6]$$

However, with the definition in Equation [4], matrix A is not invertible and no solution can be found. Matrix A can be made invertible if the following Taylor series approximation is used

$$m_j(t+f) = \sum_{l=0}^{L}\frac{f^l}{l!}\left(\frac{\partial}{\partial t}\right)^l m_j(t) \quad [7]$$

With the new matrix A becoming

$$A_{nj}(k) = \frac{1}{N}\sum_{f=0}^{N-1}\sum_{l=0}^{L}\exp\left[\frac{-2\pi i}{N}f(j-n)\right]\frac{f^l}{l!}\left(\frac{2\pi ik}{T}\right)^l \quad [8]$$

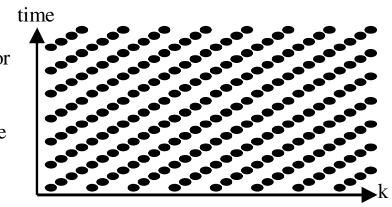

Figure 1.
Undersampling pattern used for SETS reconstruction. In this case every 4^{th} line is acquired with the pattern shifting up one line each time frame.

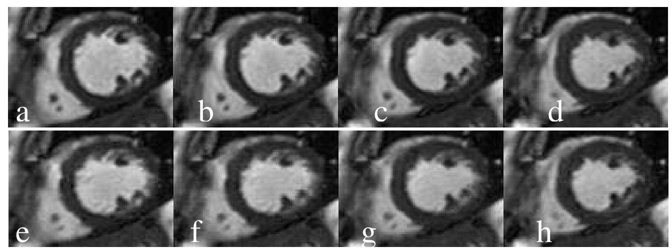

Figure 2. (a-d) Select frames of a 2D GRE cine cardiac sequence and (e-h) x8 undersampled images reconstructed using SETS

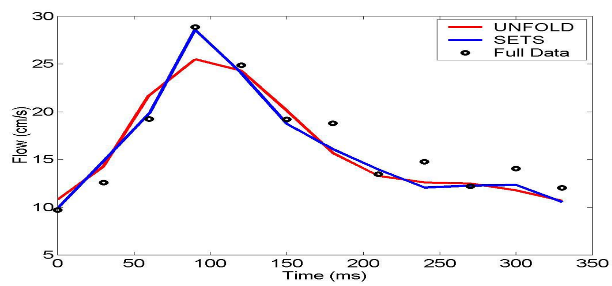

Figure 3. Phase contrast data collected from a pig with a 2D GRE sequence. Full data consists of 12 time frames and a 256x256 image size. Both the UNFOLD and SETS data were undersampled by 4.

Discussion

If L is made too large then the condition of Matrix A becomes too large and the solution becomes unstable. With UNFOLD, some prior knowledge about the bandwidth of the temporal spectrum of each pixel can be used to optimize the filtering used. With SETS, this prior knowledge can be used to assign a different value of L for each pixel in the image, making the upper limit on the summation in Equation [8] L(j) rather than L in general. Consider now an image with N_y x N_x pixels with the phase encode direction in the k_y direction and undersampled by acquiring every N^{th} line with a sampling pattern similar to Figure 1. If the object is dynamic, the signal from any pixel will alias into N_y-1 other pixels. Create a 1xN_y vector of these pixels involved in the aliasing and apply the above theory to solve for the unaliased signal in each of the N_y pixels. The process is repeated until all the pixels in the image are reconstructed.

Figure 2 shows the results of SETS applied to a GRE cine cardiac series. Fig. 2(a-d) are the images from 4 time frames (not successive) in the series reconstructed from the full data set and Fig. 2(e-h) are the same images reconstructed with undersampled data (N=8). The images reconstructed using SETS show little aliasing artifact due to the undersampling. Figure 3 shows both UNFOLD and SETS applied to flow measurement in a pig artery from 2D GRE phase contrast images. Some prior knowledge about the pixels temporal spectrum can be obtained from sliding window images. This prior knowledge was used to optimize both the UNFOLD and SETS methods. While the sliding window gives an approximation of the signal, both UNFOLD and SETS can improve on those results, with SETS being able to more accurately match the peak values in the flow velocity curves.

Conclusion

SETS is a flexible reconstruction tool for undersampled data that is similar to UNFOLD but offers improved performance in applications where there is a small degree of overlap in the temporal spectrum of aliased pixels.

References

1. B. Madore, GH Glover, NJ Pelc. UNFOLD Applied to Cardiac Imaging and fMRI. Magn Reson Med 1999;42:813-828