A general formulation for quantitative g-factor calculation in GRAPPA reconstructions

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Introduction:

Parallel Magnetic Resonance Imaging (pMRI) reconstructions such as SENSE and GRAPPA come with a loss in signal to noise ratio (SNR) compared to the non-accelerated images. The SNR is decreased by the square root of the reduction factor R and an additional factor which results in a spatially-variant noise enhancement that strongly depend on the encoding capacity of the receiver array. For SENSE–type reconstructions, this factor has been quantitatively derived and is known as geometry (g) -factor in the parallel imaging community [1]. Recently, several approaches have been described to estimate the noise enhancement in GRAPPA-type reconstructions e.g [2,3,4]. In this work, a general formulation for quantitatively calculating GRAPPA g-factors directly from the reconstruction weights for both the uncombined single coil images as well as combined images (e.g. Sum Of Squares SOS) is presented.

Theory:

It has been shown that the GRAPPA reconstruction can either be formulated in k-space or in image space (Eq. 1) [2,3]. The reconstruction weights w_{kl} are typically derived in k-space from the autocalibration data (ACS) and are then applied to the undersampled k-space signal S_l^{red} via a convolution process or directly to the folded images I_l^{red} by a multiplication with W_{kl} in image space to arrive at the final images I_k^{acc}. By exploiting the convolution theorem, the coefficients W_{kl} can be derived from the w_{kl} by inverse 2D Fourier transform.

$$(1) \quad S_k^{acc} = \sum_{l=1}^{N} S_l^{red} \otimes w_{kl} \quad \Rightarrow \quad I_k^{acc} = \sum_{l=1}^{N} W_{kl} \cdot I_l^{red}$$

The GRAPPA formulation in the image domain is useful when analyzing how the noise in the acquired reduced coil images propagates into the final accelerated image after GRAPPA reconstruction. To this end, the variance $\sigma^2(I_k^{acc})$ of the accelerated GRAPPA images must be determined. In the case of noise correlations between the receiver channels with covariance values $\sigma^2(n_k^{red},n_l^{red})=\sigma_{kl}^2$ the variance in the accelerated k-th coil image can be written using matrix $\mathbf{W}$ containing the GRAPPA weights W_{kl} and $\mathbf{\Sigma}^2$ the noise covariance σ_{kl}^2 and variance σ_{kk}^2 values:

$$(2) \quad \sigma^2\left(I_k^{acc}\right) = \sigma^2\left(n_k^{acc}\right) = \sigma^2\left(\sum_{l=1}^{N} W_{kl} \cdot n_l^{red}\right) = \left|\mathbf{W} \cdot \mathbf{\Sigma}^2 \cdot \mathbf{W}^H\right|_{kk}$$

where H denotes the transpose complex conjugate of a matrix. Taking into account that the variance of the fully encoded k-th coil image is reduced by the acceleration factor R

$$(3) \quad \sigma^2\left(I_k^{full}\right) = \sigma^2\left(n_k^{full}\right) = \frac{1}{R} \cdot \sigma_{kk}^2 = \frac{1}{R} \cdot \left|\mathbf{\Sigma}^2\right|_{kk}$$

the GRAPPA g-factor in the k-th coil can be calculated according to:

$$(4) \quad g_k = \frac{SNR_k^{full}}{SNR_k^{acc} \cdot \sqrt{R}} = \frac{\sigma\left(n_k^{acc}\right)}{\sigma\left(n_k^{full}\right) \cdot \sqrt{R}} = \frac{\sqrt{\left|\mathbf{W} \cdot \mathbf{\Sigma}^2 \cdot \mathbf{W}^H\right|_{kk}}}{\sqrt{\left|\mathbf{\Sigma}^2\right|_{kk}}}$$

Similar to the considerations above, a g-factor formulation for combined images can also be derived:

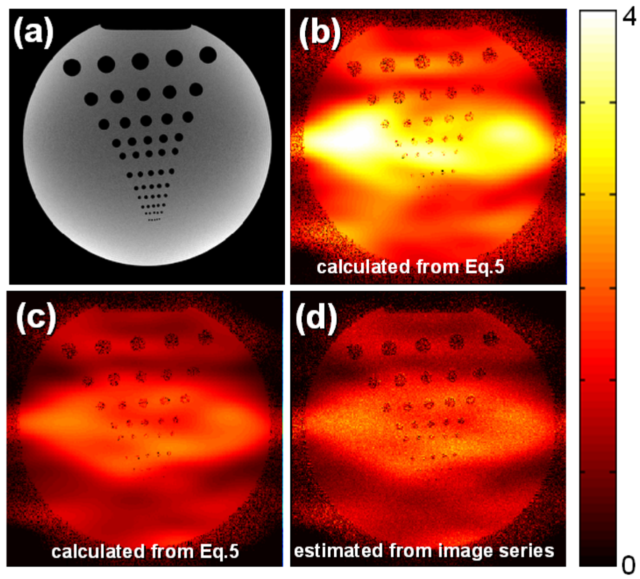

Figure 1: (a) SOS combined GRAPPA reconstruction (R=3) and corresponding quantitative GRAPPA g-factor maps derived from the GRAPPA weight set using Eq. 5 (b) without and (c) with accounting for the noise correlations. (d) Experimentally derived g-factor map from an image series as a reference.

$$(5) \quad g_{comb} = \frac{SNR_{comb}^{full}}{SNR_{comb}^{acc} \cdot \sqrt{R}} = \frac{\sqrt{\left|(p \cdot \mathbf{W}) \cdot \mathbf{\Sigma}^2 \cdot (p \cdot \mathbf{W})^H\right|}}{\sqrt{\left|(p \cdot \mathbf{1}) \cdot \mathbf{\Sigma}^2 \cdot (p \cdot \mathbf{1})^H\right|}} \quad \text{with} \quad I_{comb}^{acc} = \sum_{k=1}^{N} p_k \cdot I_k^{acc} = \sum_{k=1}^{N} p_k \cdot \sum_{l=1}^{N} W_{kl} \cdot I_l^{red}$$

The coefficients p_k in the vector p can simply be determined either from the low resolution ACS data or the high resolution accelerated GRAPPA images. In the case of a SOS reconstruction, these coefficients are given by $p_k=I^*_k/I_{SOS}$.

Material and Methods:

In order to provide an accurate reference, 500 fully-encoded phantom images were acquired with identical parameters (TE/TR=7.1ms/40ms, α=30°, bw=100Hz/px, FOV=210x210mm2, matrix=256x256) on a 1.5T clinical Avanto Scanner equipped with a standard 12 channel head coil (Siemens Medical Solutions, Erlangen, Germany). In order to allow for a correction of the noise correlations between the individual channels, an additional noise-only image was acquired using identical imaging parameters. Various accelerated (R=2,3,4) GRAPPA reconstructions were performed for each image in the series. The reference g-factors were determined experimentally by calculating the mean and standard deviation in each image pixel throughout the series for the accelerated and unaccelerated case for the single coil images, SOS and normalized adaptively combined images [5] according to Eq. 4 and 5, respectively.

Results:

Fig. 1 shows (a) the SOS-combined (R=3) GRAPPA reconstruction and a comparison between calculated (directly form a single weight set) SOS g-factor maps (b) without and (c) with accounting for the noise correlations. In (d) the experimentally determined reference (from the image series) is displayed. As can be seen, the GRAPPA g-factor calculation presented here (d) shows an excellent agreement with the experimentally determined g-factor maps (c) and therefore provides a fast and easy tool to quantify the noise enhancement in GRAPPA-type reconstructions. However, the images also demonstrate the importance of including the noise correlation to provide accurate results.

Conclusion:

The g-factor calculation for GRAPPA reconstructions presented here allows one to quickly and accurately estimate the geometry related non-uniform noise enhancement in GRAPPA reconstructions directly from the GRAPPA reconstruction weight set. This enables, for example, a fast comparison between the performances of different reconstruction kernels or SENSE reconstructions. In addition, since g-factor maps for the uncombined GRAPPA images are available, they can potentially be used to find an SNR-optimized signal combination.

References:

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