A novel, accurate method for computing the density compensation function for regridding non-Cartesian k-space data

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Introduction: Regridding [1] is a popular technique for reconstructing non-Cartesian k-space data. The k-space data has to be multiplied by a density compensation function (DCF) to account for varying sampling density, prior to convolution and resampling on a Cartesian grid. For some trajectories like radial and Archimedean spiral, the DCF can be determined analytically but in general it is approximated by the convolution of the sampling grid with the kernel itself [1], using Voronoi grids [2] or determined iteratively [3]. We propose a new, accurate method for computing the DCF. Starting with an impulse in spatial domain whose k-space is unity everywhere, we constrain the density function to ensure that the reconstructed image approximates an impulse as closely possible.

Theory: For brevity and clarity, we describe the 2X regridding operation using matrix notation (matrices are in caps and vectors in small). If μ is the non-Cartesian k-space vector (row ordered), the regridded image vector $\boldsymbol{i}$ and the residual aliasing term $\boldsymbol{e}$ can be written as

$$i + e = DFCW\mu \;\Rightarrow\; ED^{-1}(i+e) = EFCW\mu \qquad (1)$$

where $\boldsymbol{W}$ the desired density compensation (diagonal) matrix, $\boldsymbol{C}$ the kernel convolution matrix, F the FFT Kronecker product, D the de-apodization matrix to null the effects of the convolution kernel, and E a matrix of 1s and 0s to crop out the central region of the 2X reconstructed image. We seek to reconstruct an impulse image whose k-space is unity ($\mu = \underline{1}$) everywhere in the region of support (ROS) of sampled k-space. Eqn (1) reduces to

$$EFCw = ED^{-1}(\hat{\delta} + e) = ED^{-1}\hat{\delta} + ED^{-1}e \qquad (2)$$

where $\hat{\delta}$ represents the convolution of an ideal impulse with the FT of the ROS of sampled k-space and w contains the diagonal elements of W. For example, in the case of spiral sampling, the ROS of sampled k-space is a disk of radius k_{max}. If we assume minimal side-lobe aliasing due to the convolution kernel in the central region (ensured by choosing a Kaiser-Bessel window [1]) and adequate suppression of aliases in outer region by the apodization matrix D^{-1}, an assumption true for a 2X resampling grid, the second term in (2) can be neglected.

$$FCw \approx D^{-1}\hat{\delta} \quad \Rightarrow \quad Cw \approx F^{-1}\{D^{-1}\hat{\delta}\} = \hat{r} \qquad (3)$$

This equation can be easily solved using iterative least squares as C is a sparse matrix. Using the same assumptions as above, it can be shown that the regirdding reconstruction algorithm is linear, shift invariant within the FOV and the result for an impulse extended to arbitrary images. In summary, starting from non-Cartesian k-space data which is unity everywhere and constraining the Cartesian regridded k-space to be unity within the sampled ROS, the density compensation function can be easily computed, taking into account the effects of the convolution kernel and deapodization.

Materials and methods: All methods were implemented on MATLAB. A synthetic phantom was sampled using radial, spiral and variable density Cartesian trajectories and data reconstructed using 3 different DCFs: convolution of sampling grid with kernel as in [1] (Jackson), iterative method of [3] (Pipe) and our method. Regridding was implemented efficiently using sparse matrices with a Kaiser-Bessel window of width 2.5 and β 11.5 used for all 3 DCFs. Note that the convolution kernel has not been optimized iteratively as in [3]. Phantom data acquired using a spiral pulse sequence (3096 pts/16 interleaves) from a 1.5T GE Excite scanner was reconstructed using the 3 different DCFs and compared.

Results: The DCFs computed for spiral trajectories using the three methods are shown in Fig. 1 in log scale for a single interleaf. Note the deviations close to the center and periphery of k-space. Fig. 2 compares profiles through the "comb" structure of the phantom scanned using a spiral trajectory. The proposed DCF (red) yields a sharper profile than the other two (blue, pink). The FFT of the reconstructed impulse using our proposed DCF is shown in Fig. 3, which is uniformly close to unity across the ROS of sampled k-space, agreeing with the theory. Fig. 4-6 shows the reconstruction error images for spiral sampling. Note the absence of high frequency and bias error terms using our DCF (Fig. 6) compared to the other 2 DCFs (Fig. 4-5). The minimum mean square error (MMSE) between the original and reconstructed image is shown in Table 1 for spiral and radial sampling. Our proposed DCF method yields superior results even in noisy cases where there is a slight degradation in SNR.

Conclusions: We have demonstrated a new accurate method for computing the DCF for arbitrary k-space sampling, which yields improved reconstructions. There is some similarity to the iterative method of [3] but our optimization criterion was to preserve the fidelity of the regridded Cartesian data rather than the non-Cartesian points that was the focus of [3]. A comparison of the FT of the reconstructed impulse for the other 2 DCFs reveals reduction in weighting of the central and outer k-space points resulting in bias and loss of resolution, which is avoided using our DCF.

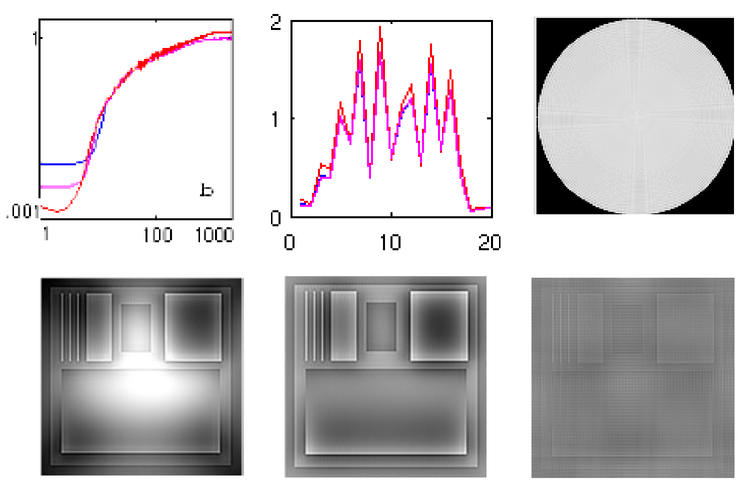

Figure order

1	2	3
4	5	6

Table 1: comparison of MSE for the 3 DCFs

MMSE ($x10^{-4}$)	Jackson Spiral/Radial	Pipe Spiral/Radial	Proposed Spiral/Radial
Noiseless	12 / 5.9	6.1 / 6.2	1.2 / 3.4
W/ noise (SNR 40 dB)	13 / 8.6	7.8 / 8.8	5.8 / 8.18

Fig 1: DCF comparison: Jackson (blue), Pipe (pink), Proposed (red)
Fig 2: Comparison of profiles through the "comb" for the 3 DCFs Jackson (blue), Pipe (pink), Proposed (red)
Fig 3: FT of the reconstructed impulse using our proposed DCF.
Fig 4-6: Reconstruction error for the 3 DCFs above (-0.3-0.3 scale original image scale [0-5])

References: 1. Jackson et al. IEEE TMI 10:473-478 (1991)
2. Rasche et al. IEEE TMI 18 :385-392 (1999)
3. Pipe et al. MRM 41 : 179-186 (1999)