An Analysis of Reciprocity in Parallel MRI

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Introduction: Parallel transmit (Tx) research began with an idea that is conceptually analogous to that of parallel receive (Rx) (1,2). Recent studies suggested that parallel Tx offers significant opportunities for accelerating multi-dimensional selective excitation and reducing SAR (3,4), and that a reciprocity relationship exists at a fundamental level between parallel Tx MRI and parallel Rx MRI (5). With the expectation that fully establishing the reciprocity relationship shall accelerate parallel Tx research and its transition to high B_0 field clinical MRI applications, this work focuses on developing results of theoretical / practical interest and exploring their implications on parallel MRI.

Methods and Results: We will compare parallel Tx and parallel Rx in a 2D Cartesian trajectory case. Such a comparison facilitates an intended conceptual depiction, and meanwhile, offers results directly applicable to an important class of parallel MRI methods. For definiteness we designate the k-space traversed by the 2D Cartesian trajectory as the k_x-k_y plane, with k_x denoting the slow traversing direction and, Δ_{kx}, the sampling period. We further designate the extreme x-coordinates of the object boundary as x_{min} and x_{max}. For the x dimension then, sampling theorem institutes the classic field-of-view (FOV) requirement imposed on the sampling period: $1/\Delta_{kx} \geq (x_{max}-x_{min})$, which originated from the need to avoid (Rx) fold-over and (Tx) sidelobe effects. This requirement is relaxed by a factor equal to the number of Rx and Tx channels with the use of, respectively, parallel Rx MRI and parallel Tx MRI. We assume the number of channels or coils is N in both the parallel Tx and parallel Rx cases.

Before elaborating on the details, we summarize the present parallel Rx image reconstruction and parallel Tx profile creation processes as follows:

reconstructed image $\hat{M}(\mathbf{r}) = \sum_{n=1}^{N} h_r^{(n)}(\mathbf{r}) \left[M(\mathbf{r}) B_{1-}^{(n)}(\mathbf{r}) \right]_{\text{aliased}}$

created excitation profile $\hat{u}(\mathbf{r}) = \sum_{n=1}^{N} B_{1+}^{(n)}(\mathbf{r}) \left[u(\mathbf{r}) h_t^{(n)}(\mathbf{r}) \right]_{\text{aliased}}$

$M(\mathbf{r})$ = magnetization profile; $B_{1-}^{(n)}(\mathbf{r})$ = nth Rx coil's B_1^- field (i.e., sensitivity) profile; $u(\mathbf{r})$ = target excitation profile; $B_{1+}^{(n)}(\mathbf{r})$ = nth Tx coil's B_1^+ field profile; $[\ldots]_{\text{aliased}}$ = aliased and bandwidth-limited version resulting from k-space sampling.

In parallel Rx, SENSE reconstruction can in general be shown to be equivalent to estimating a composite image with the sum of N optimally weighted aliased individual channel images (each from a 2DFT, with a period of $1/\Delta_{kx}$ along x over the full FOV): $\hat{M}(x,y) = \sum_{n=1}^{N} h_r^{(n)}(x,y) \tilde{M}^{(n)}(x,y)$ Or in terms of image pixels, $[\hat{M}(p_1\Delta_x, p_2\Delta_y) \cdots \hat{M}((p_1+mL)\Delta_x, p_2\Delta_y) \cdots]^T = \mathbf{H}_r^T [\tilde{M}^{(1)}(p_1\Delta_x, p_2\Delta_y) \cdots \tilde{M}^{(N)}(p_1\Delta_x, p_2\Delta_y)]^T$, where m is the index for a set of aliased pixel locations, $\Delta_x = 1/\Delta_{kx}/L$, $\mathbf{H}_r = [\mathbf{h}_{r,1} \ldots \mathbf{h}_{r,m} \ldots]$, and vector $\mathbf{h}_{r,m}$ represents $[h_r^{(1)}((p_1+mL)\Delta_x, p_2\Delta_y) \ldots h_r^{(N)}((p_1+mL)\Delta_x, p_2\Delta_y) \ldots]^T$. Fulfilling the requirements that the estimated composite image is a) un-biased or aliasing-free, and b) of minimum variance or lowest noise, translates into a constrained optimization, which determines the spatial weighting functions and the final image for one set of aliased pixel locations at a time:

$$\begin{aligned} &\text{minimize} \quad \mathbf{h}_{r,m}^{*} \mathbf{\Psi}^T \mathbf{h}_{r,m} \\ &\text{subject to} \quad \mathbf{S}^T \mathbf{h}_{r,m} = \mathbf{e}_m \end{aligned} \qquad [1]$$

In Eq.1 $\mathbf{S}$ and $\mathbf{\Psi}$ are, respectively, the sensitivity and noise correlation matrices (1), and $\mathbf{e}_m$ has a 1 at its mth entry and 0's elsewhere. The constrained optimization is solved by $\mathbf{h}_{r,m} = ((\mathbf{S}^*\mathbf{\Psi}^{-1}\mathbf{S})^{-1}\mathbf{S}^*\mathbf{\Psi}^{-1})^T \mathbf{e}_m$. Image reconstruction using $\mathbf{H}_r$ is thus readily confirmed as identical to SENSE, and with the same pixel noise covariance of $(\mathbf{S}^*\mathbf{\Psi}^{-1}\mathbf{S})^{-1}$.

In parallel Tx, recent studies indicated that design of excitation pulses in the small flip regime can be advantageously formulated as a constrained optimization as well, which minimizes SAR while forcing aliasing sidelobe suppression (4,5). This formulation can be shown to be equivalent to calculating the parallel RF pulses' image-space representations by forming optimally weighted versions of the target excitation profile: $[f^{(1)}(p_1\Delta_x, p_2\Delta_y) \cdots f^{(N)}(p_1\Delta_x, p_2\Delta_y)]^T = \mathbf{H}_t [u(p_1\Delta_x, p_2\Delta_y) \cdots u((p_1+mL)\Delta_x, p_2\Delta_y) \cdots]^T$, i.e., $\mathbf{f} = \mathbf{H}_t \mathbf{u}$, where the $f^{(n)}$'s are the image-space periodic patterns directly related to the parallel RF pulses by 2DFT (5), u is the target excitation profile, $\mathbf{H}_t = [\mathbf{h}_{t,1} \ldots \mathbf{h}_{t,m} \ldots]$, and vector $\mathbf{h}_{t,m}$ represents $[h_t^{(1)}((p_1+mL)\Delta_x, p_2\Delta_y) \ldots h_t^{(N)}((p_1+mL)\Delta_x, p_2\Delta_y) \ldots]^T$. The calculation of the $f^{(n)}$'s is carried out for one set of aliased locations at a time:

$$\begin{aligned} &\text{minimize} \quad \mathbf{h}_{t,m}^{*} \mathbf{\Phi}\, \mathbf{h}_{t,m} \\ &\text{subject to} \quad \mathbf{C}\, \mathbf{h}_{t,m} = \mathbf{e}_m \end{aligned} \qquad [2]$$

In Eq.2 $\mathbf{C}$ and $\mathbf{\Phi}$ are, respectively, the B_1^+ and power correlation matrices (5). The constrained optimization is solved by $\mathbf{h}_{t,m} = \mathbf{\Phi}^{-1}\mathbf{C}^*(\mathbf{C}\,\mathbf{\Phi}^{-1}\mathbf{C}^*)^{-1}\mathbf{e}_m$, hence $\mathbf{f} = \mathbf{H}_t \mathbf{u} = \mathbf{\Phi}^{-1}\mathbf{C}^*(\mathbf{C}\,\mathbf{\Phi}^{-1}\mathbf{C}^*)^{-1}\mathbf{u}$. 2DFT's of the $f^{(n)}$'s then lead to the RF pulse waveforms. This formulation gives identical results to that of (5) in terms of RF pulse waveforms and SAR.

The symmetry readily identified in the results above offers unique insights on parallel MRI. Among the significant ones, we note that first, matrices $\mathbf{S}^T$ and $\mathbf{C}$ are structurally identical though they carry entries that correspond to transverse B_1 fields' B_1^- and B_1^+ components respectively (6). In cases where sample loss dominates, both $\mathbf{\Psi}$ and $\mathbf{\Phi}$ originate primarily from random motion of charges in the sample, and, by principle of reciprocity (6), assume values due to the same type of E field correlations (7). Second, replacing $\mathbf{\Psi}$ with $\mathbf{\Phi}^T$, and $\mathbf{S}$ with $\mathbf{C}^T$, the spatial weights from the Eq.1-based alternative SENSE reconstruction can be readily used to derive parallel RF pulse waveforms, exemplifying a tactic where one morphs a parallel Rx technique for parallel Tx use, and vice versa. This property was verified in our preliminary phantom imaging experiments. Third, in producing uniform excitation (at high B_0 field) with tailored RF pulses, the illustrated parallel Tx method represents a distributed approach to RF excitation, promising SAR characteristic improvements over traditional volume coil Tx, as analogous to SNR characteristic improvements with phased-array / parallel Rx methods over volume coil Rx (7,1,2). The analogy for example, manifests in the symmetry between the g-factor (2) and the g_t-factor (5), which characterize, respectively, geometrical dependency of image SNR in parallel Rx MRI and overall SAR in parallel Tx MRI:

$$g = \left[\mathbf{e}_m^* \left(\mathbf{S}^*\mathbf{\Psi}^{-1}\mathbf{S} \right)^{-1} \mathbf{e}_m \Big/ \mathbf{e}_m^* \left(\mathrm{DIAG}\left(\mathbf{S}^*\mathbf{\Psi}^{-1}\mathbf{S} \right) \right)^{-1} \mathbf{e}_m \right]^{1/2} \quad \text{and} \quad g_t = \left[\boldsymbol{\mu}^* \left(\mathbf{C}\,\mathbf{\Phi}^{-1}\mathbf{C}^* \right)^{-1} \boldsymbol{\mu} \Big/ \boldsymbol{\mu}^* \left(\mathrm{DIAG}\left(\mathbf{C}\,\mathbf{\Phi}^{-1}\mathbf{C}^* \right) \right)^{-1} \boldsymbol{\mu} \right]^{1/2}$$

However, the spatial variation of parallel Rx image noise, due to non-unitary image reconstruction, is not to be compared with parallel Tx's RF power deposition pattern. The analogy, and an understanding that parallel Tx's goal of maximizing flip-angle to RF-power ration mirrors parallel Rx's goal of maximizing signal to noise ratio, have led to the new concept of ultimate intrinsic SAR, which underlies our current investigation on the intrinsic (i.e., without any assumption on Tx coil configuration) lower bound of RF power deposition. Fourth, $(\mathbf{C}\mathbf{\Phi}^{-1}\mathbf{C}^*)^{-1} = \mathbf{H}_t^*\,\mathbf{\Phi}\,\mathbf{H}_t$, and probably of further interest, hypothetically interchanging {coil B_1^+ patterns, $\mathbf{\Phi}$} with {designed h_t weightings, $\mathbf{\Phi}^{-1}$} leads to the same parallel Tx outcome. An analogous result exists in the parallel Rx case. Such duality relationships may possibly lead to new perspectives useful to coil design / optimization.

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