Unified Mathematical Model of q-space and diffusion tensor imaging

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Introduction:

Statistical imaging of random H_2O motions using MRI has followed two mathematical paradigms: **diffusion-weighting** and ***q*-space** analyses. In diffusion-weighted analyses, the ensemble of molecules is assumed to evolve with a (not necessarily isotropic) Gaussian probability density/point spread function (PSF)—or a mixture of such densities—and the data analysis is oriented towards determining the parameters of the PSF (e.g., the apparent diffusion tensor). In *q*-space analyses, the transport is allowed to have a quite general PSF; as a result, much more data is required to reconstruct the PSF. The two analytical paradigms come from two different experimental communities with little overlap, since diffusion-weighted imaging is practicable in humans, whereas accurate *q*-space imaging is not. Our goal is to generalize the analysis of [1], which relates *q*-space and diffusion weighting.

Analysis:

We denote by $M(\mathbf{x},t)$ the transverse magnetization and by $\hat{M}(\mathbf{q},t)$ its spatial Fourier transform. In the absence of magnetic field gradients, we model the transport of magnetization from its initial state $M(\mathbf{x},0)$ to time t by convolution (*) with an unknown time-dependent PSF $P(\mathbf{x},t)$:

$$M(\mathbf{x},0) \xrightarrow[t]{} M(\mathbf{x},0) * P(\mathbf{x},t) \;\Rightarrow\; \hat{M}(\mathbf{q},0) \xrightarrow[t]{} \hat{M}(\mathbf{q},0)\cdot \hat{P}(\mathbf{q},t) \;\Rightarrow\; \frac{\partial \hat{M}(\mathbf{q},t)}{\partial t} = \frac{\partial \hat{P}(\mathbf{q},t)/\partial t}{\hat{P}(\mathbf{q},t)} \cdot \hat{M}(\mathbf{q},t)$$

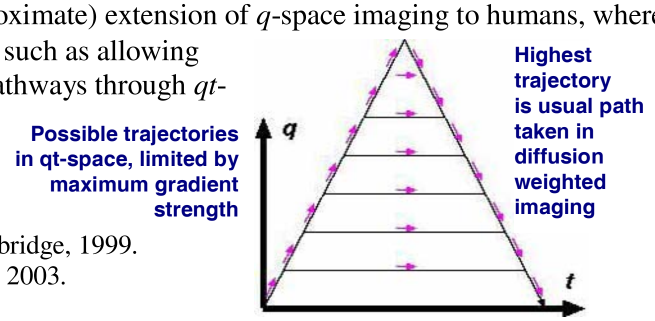
Define $\hat{P}(\mathbf{q},t) = e^{-u(\mathbf{q},t)}$ (for diffusion, $u(\mathbf{q},t) = \mathbf{q}\cdot\mathbf{D}\cdot\mathbf{q}\,t$, where $\mathbf{D}$ is the diffusion tensor to be estimated), so that $\hat{P}(\mathbf{q},t)^{-1}\cdot \partial\hat{P}(\mathbf{q},t)/\partial t = -\partial u(\mathbf{q},t)/\partial t \equiv -u_t(\mathbf{q},t)$. With gradients, $d\mathbf{q}/dt = \gamma \mathbf{G}(t)$, and the magnetization transport model is:

$$\frac{\partial M(\mathbf{x},t)}{dt} = -i\frac{d\mathbf{q}}{dt}\cdot \mathbf{x}\, M(x,t) + \Im^{-1}_{\mathbf{q}\leftrightarrow\mathbf{x}}\left[-u_t(\mathbf{q},t)\hat{M}(\mathbf{q},t)\right] \;\Rightarrow\; \frac{\partial \hat{M}(\mathbf{q},t)}{\partial t} - \frac{d\mathbf{q}}{dt}\cdot\nabla_{\mathbf{q}}\hat{M}(\mathbf{q},t) = -u_t(\mathbf{q},t)\hat{M}(\mathbf{q},t)$$

The last equation is a first order PDE in $(\mathbf{q},t)$ space, which can be solved using the method of characteristics: $\hat{M}\big(\mathbf{q}_0 - \mathbf{q}(t),t\big) = e^{-\int_0^t u_t(\mathbf{q}(\tau),\tau)\,d\tau}\hat{M}(\mathbf{q}_0,0)$, where $\mathbf{q}_0$ is an arbitrary vector in q-space. For imaging purposes, the trajectory $\mathbf{q}(t)$ is rewound to $\mathbf{q} = \mathbf{0}$ at some time *T* before the *k*-space readout begins. Assuming that the *k*-space region covered is small enough not to induce significant diffusive effects itself, then we find that the attenuation of the image is given by $E = e^{-\int_0^T u_t(\mathbf{q}(t),t)\,dt}$ or $-\ln(E) = \int_0^T u_t(\mathbf{q}(t),t)\,dt = -\int_0^T \left[\hat{P}(\mathbf{q}(t),t)^{-1}\cdot\partial\hat{P}\big(\mathbf{q}(t),t\big)\big/\partial t\right]dt$; that is, a general trajectory through *qt*-space gives a tomographic result about the time evolution of the Fourier transform of the PSF for water transport.

Discussion: qt-Space Tomography:

In the diffusion limit, $\ln(E) = -\int_0^T \mathbf{q}(t)\cdot\mathbf{D}\cdot\mathbf{q}(t)\,dt$, which is the basis for estimating the diffusion tensor **D** by using different paths through *qt*-space. In the *q*-space PGSE experiment, $T = \Delta$ and $\mathbf{q}(t) = \text{const}$ (since the pulsed gradient duration δ is assumed small), yielding $E(\mathbf{q}) = e^{-\int_0^\Delta u_t(\mathbf{q},t)\,dt} = e^{-u(\mathbf{q},\Delta)} = \hat{P}(\mathbf{q},\Delta)$, the usual *q*-space imaging attenuation. If δ is *not* small, then $\mathbf{q}(t) \neq \text{const}$ and the relationship between E and $\hat{P}(\mathbf{q},t)$ is more complicated and involves all intermediate times $0 \le t \le \Delta$. If enough different trajectories through *qt*-space were traversed, *and* a parameterized mathematical model for $\hat{P}(\mathbf{q},t)$ adopted (cf. [2,3]), then the parameters of $\hat{P}(\mathbf{q},t)$ could be estimated from the $-\ln(E)$ measurements. Such a technique may allow a systematic (if approximate) extension of *q*-space imaging to humans, where δ cannot be small. Various extensions of this theory are possible, such as allowing for transport effects during excitation and for multiple coherent pathways through *qt*-space created by multiple RF pulses [4].



References:

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