Necessary and sufficient conditions for the admissibility of DTI Gradient Vectors

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Introduction:

Diffusion tensor imaging [1] is one of the most important diagnostic tools to day (see [4] for a state of the art review). Earlier, it was generally believed that the diffusion tensor and its eigenvalues can be determined by diffusion-weighted MRI with the diffusion gradients applied in 'at least six non-collinear directions' (see e.g., [2], [3]). Subsequently some necessary conditions, namely, 'There must be at least one set of six gradient vectors with the following properties: (1) no two gradient vectors are parallel or antiparallel; (2) if three vectors belong to a 2D subspace then the remaining three vectors must be linearly independent; (3) no subset of four gradient vectors may belong to the same 2D subspace,' for the same were found in [5]. However, these conditions are far from also being sufficient.

As an example, the n vectors $g_j = (\cos(2\pi j/n)\sin\phi, \sin(2\pi j/n)\sin\phi, \cos\phi)'$, $0 \le j \le n-1$, for whatever $0 < \phi < \pi/2$ and whatever $n \ge 6$, satisfy all the conditions listed above, but they are not good enough for the tensor calculation.

The purpose of this presentation is to rigorously establish a set of 'necessary and sufficient conditions' on the diffusion gradients for the determinacy of the diffusion tensor as follows.

Material and Methods:

In DTI one utilizes a set of m diffusion gradient direction vectors $g_i = (x_i, y_i, z_i)'$, $1 \le i \le m$, say, and after subtracting the logarithms of the associated diffused signal intensities from the logarithm of the diffusion free signal from the same voxel and dividing the result by the gradient strength factor 'b' one gets the data d_i occuring on the right hand side of the equations:

$$D_{xx}\, x_i^2 + D_{yy}\, y_i^2 + D_{zz}\, z_i^2 + 2\, D_{yz}\, y_i z_i + 2\, D_{zx}\, z_i x_i + 2\, D_{xy}\, x_i y_i = d_i,\ 1 \le i \le m,$$

where $g_i = (x_i, y_i, z_i)'$, $1 \le i \le m$, are the diffusion gradient direction vectors (assumed to be unit vectors) and

$$\begin{pmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{xy} & D_{yy} & D_{yz} \\ D_{xz} & D_{yz} & D_{zz} \end{pmatrix} = D$$

describes the required diffusion tensor for the voxel. The computation of D follows a least squares solution of the above system of m-equations. The necessary and sufficient condition for the determination of D is called '**admissibility of DTI Gradient Vectors**' is that the DTI design matrix X

$$X \equiv \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{bmatrix} \equiv \begin{bmatrix} x_1^2 & y_1^2 & z_1^2 & y_1 z_1 & z_1 x_1 & x_1 y_1 \\ x_2^2 & y_2^2 & z_2^2 & y_2 z_2 & z_2 x_2 & x_2 y_2 \\ .. & .. & .. & .. & .. & .. \\ x_m^2 & y_m^2 & z_m^2 & y_m z_m & z_m x_m & x_m y_m \end{bmatrix} \equiv \begin{bmatrix} w_1 \\ w_2 \\ .. \\ w_m \end{bmatrix}$$

have rank six, which is equivalent to its column vectors v_i, $1 \le I \le 6$ being linearly independent, or , equivalently that at least one subset of some six of its row vectors w_i, $1 \le I \le m$, be linearly independent. The presentation is to give a rigorous mathematical proof of our following result giving necessary and sufficient conditions on the gradient vectors for their admissibility. The inadmissible diffusion gradient cases are shown in Fig. 1:

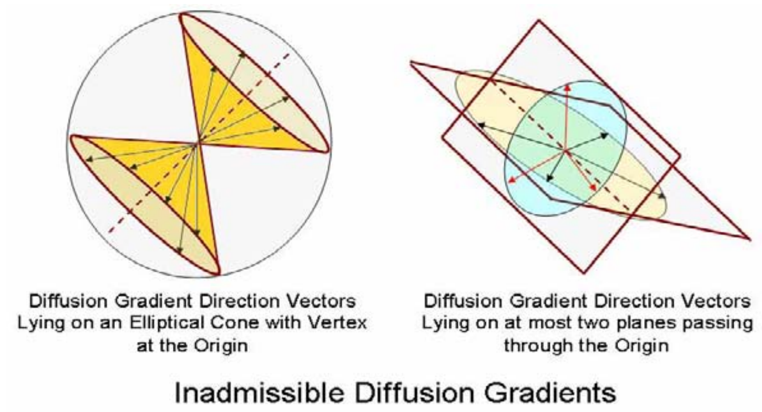

Fig. 1

Results:

Theorem (Admissibility of DTI Gradient Vectors). The gradient vectors in DTI are admissible iff there hold: (A) the vectors do not lie in less than three planes through origin, and (B) the vectors do not lie on an elliptical cone with vertex at origin.

We will also show how the 'necesaary conditions' of Özcan could be deduced as a corollary of the above 'necessary and sufficient conditions.'

Discussion:

The above admissibility criterion provides a simple geometiric visualization of admissible DTI gradient vector sets, subject to which one could exercise the rest of the freedom to construct such a set appropriate to a particular experimental set up.

Reference:

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