Equivalence of Fourier and oSVD Deconvolution in Dynamic Perfusion Measurements: Mutual Filter Transform

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Introduction

Dynamic susceptibility contrast measurements are processed by deconvolution of the residue Function R as defined in equation (1) [1]. The deconvolution is commonly performed using the Fourier convolution theorem or the singular value decomposition (SVD). A circular interpretation of the convolution leads to the circular SVD (oSVD) that is mathematically equivalent to the Fourier deconvolution approach [2]. Despite this fact the comparison of the impact of those methods on the estimate of perfusion parameters was obscure, as different filters where applied in the different approaches for noise reduction. In this work we present how oSVD and Fourier filters can be transformed into each other, by analyzing the above mentioned mathematical equivalence.

Theory

The dynamic perfusion equation (1) arises from the tracer dilution theory [1]. The measured tissue concentration $c_t(t)$ of an injected contrast agent is computed from an arterial input function a(t) by the convolution:

$$c_t(t) = CBF \cdot R(t) * a(t) \qquad (1)$$

where R is the residue function and CBF the blood flow. The discretization of equation (1) involves an extrapolation of the measured curves (e.g. zero filling) or periodic (circular) interpretation of the convolution in order to avoid range violations. The Fourier convolution theorem can be applied if the measured curves are either zero filled to meet the integration limits in the transform or interpreted as if they reoccur periodically (circular interpretation). At increasing size of the zero filling, the deconvolution result converges to the one obtained by periodic interpretation. Facing this fact and that zero filling strongly decreases the computational efficiency; we focus on the circular interpretation in this work. Circular interpretation also leads to an independence of the estimated flow on the arrival time [3]. Equation (1) can be rewritten in matrix form for this case:

$$\vec{c} = A \cdot \vec{R} \quad (2) \qquad \text{where} \qquad A_{ij} = \begin{cases} a(t_j - t_i) & \text{if} \quad i \le j \\ a(N \cdot \Delta t + (t_j - t_i)) & \phantom{\text{if}} \quad i > j \end{cases} \quad (3)$$

The matrix A can be inverted by diagonalization that can be achieved by means of SVD or Fourier decomposition, yielding a vector that represents the residue function R:

SVD: $\vec{R} = V \cdot S^{-1} \cdot U^T \cdot A \cdot \vec{c}$ (4) Fourier: $\vec{R} = F \cdot S^{-1} \cdot F^{-1} \cdot A \cdot \vec{c}$ (5) with $F^{-1} \cdot S^{-1} \cdot F = A \cdot A^T = U \cdot S \cdot V^T$ (6)

The matrices U,V and the diagonal matrix S in equation (4) are obtained by the SVD algorithm [4] for the matrix $B=AA^T$. The columns of the matrix F in equation (5) contain the Fourier basis as vectors. If the Fourier convolution theorem holds, F diagonalizes B as well. If a matrix can be diagonalized by two other matrices, they have to be same up to a permutation of their columns containing the eigenvectors of the matrix. This is the reason for the mathematical equivalence of oSVD and Fourier deconvolution. The corresponding permutation matrix can be found by the property, that the SVD algorithm sorts the eigenvalues in descending order whereas the Fourier approach leads to an order with respect to the frequency. Thus the matrix V is only a column permutation of V. Such a transform between the two perspectives can easily be found for all subspaces, where the eigenvalues are large enough to be sorted unambiguously in descending order. The corresponding permutation matrix can then be applied to filters designed for a particular deconvolution approach.

Results

The eigenvalues found by the two methods are plotted in Fig. 1. The values differ by less then 10^{-14} (in MATLAB) which underlines the mathematical equivalence. The reordering of the Fourier spectrum in the oSVD spectrum explains the pairwise appearance of eigenvalues, as real signals have a symmetric spectrum. The eigenvectors found by oSVD as columns of the matrix V are the Fourier basis functions as shown in Fig. 2.

For a monotonously decreasing Fourier spectrum of the concentration curves the permutation is unity. This case is met if the first pass of the contrast agent injection has been extracted prior to the deconvolution [5]. If recirculation boli are included, a side peak in the Fourier spectrum appears at the corresponding frequency. In this case the permutation can be found by the reordering explained above. The filter transform properties are depicted in Fig. 3 and Fig. 4. A threshold filter applied to the oSVD eigenvalues (blue bar in Fig. 3) is split up into a band pass filter in the frequency domain as depicted in Fig. 4 if the spectrum shows side peaks with amplitudes above the oSVD threshold.

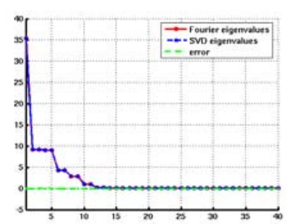

Fig. 1: *Eigenvalues found by SVD and Fourier decomposition in descending order.*

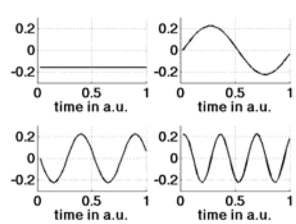

Fig. 2: *Plot of the eigenfunctions found by the SVD deconvolution sorted by frequency.*

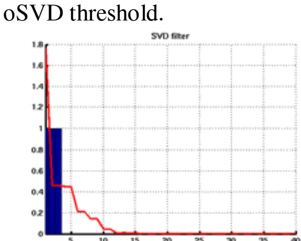

Fig. 3: *SVD eigenvalues. The underlying block indicates the pass interval of the eigenvalues.*

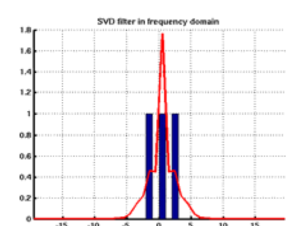

Fig. 4*: Fourier spectrum. The underlying blocks indicate the pass frequency range filter in Fig. 3.*

Discussion

The presented work gives insight in the properties of SVD and Fourier deconvolution. If only the first bolus passage is contained in the measured concentration time courses, the filter transform is the unity matrix. Recirculation yields symmetric side lobes in the Fourier spectrum that cause the pass interval of typical SVD threshold filters to distribute to the peaks in the spectrum. A similar finding was also reported in [6]. Using the presented transform, the knowledge on filter design from both methods can be used in order to come up with better filters for perfusion processing.

References

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