block circulant band diagonal property for parallel imaging reconstruction in cartesian k-space

B. Kressler[1]**, T. Liu**[1]**, K. Wang**[1]**, and Y. Wang**[2]

[1]Cornell University, [2]Radiology, Cornell University, New York, New York, United States

Introduction

The two most commonly used parallel imaging techniques in MRI are 1) generalized SMASH or Generalized Autocalibrating Partially Parallel Acquisitions (GRAPPA) [1] and 2) SENSitivity Encoding (SENSE) [2]. While the SENSE recon is mathematically rigorous, the GRAPPA recon makes physical assumptions that may require rigorous justifications. In this work, we investigate the structure of the SENSE matrix in k-space and show that although the coefficients in SENSE and GRAPPA matrices are derived from solutions to very different systems of equations, the matrices possess closely related structures as block circulant band diagonal matrices. This structure may serve as a rigorous clarification for GRAPPA recon.

Theory

Existence of block circulant band diagonal reconstruction matrix as from SENSE. The MR imaging acquisition process can be realized as a convolution operation between coil sensitivities and the excited spin distribution in k-space [3]: S = CF, where F represents full k-space data of the excited spin distribution, S the acquired data, and C the coil matrix. For a coil matrix of smooth spatial distribution, C is block circulant band diagonal in Cartesian sampling. The SENSE solution is $F = C^+S$, where C^+ is the Moore-Penrose pseudoinverse of the coil matrix, or in coil specific recon $F_j = (D_jC^+)S$, where Dj is the jth coil matrix (circulant band diagonal). Because the product and inverse of block circulant band diagonal matrices are also block circulant band diagonal matrices, the SENSE reconstruction matrix in k-space is block circulant band diagonal.

Justification for GRAPPA. The GRAPPA recon essentially assumes the existence of block circulant band diagonal reconstruction matrix. The band diagonal property allows calibration using a few lines in the k-space center, and the block circulant property allows data interpolation at other locations in k-space. Accordingly, GRAPPA recon can be formulated as $F_j = G_jS$, where Gj is a block circulant block diagonal matrix calibrated at the k-space center.

Differences between GRAPPA and SENSE. Where the recon matrices for both SENSE (in k-space) and GRAPPA are of similar form, GRAPPA is constrained to fitting recon matrix parameters to measured autocalibration lines (points) in k-space, and SENSE is constrained to fitting matrix parameters to all measured k-lines (least errors). Accordingly, in the case of N_c coil elements with reduction factor $R=N_c$, SENSE and GRAPPA recon matrix would look similar, otherwise for $R<N_{c,}$ they would look different, and the general recon matrix is not unique.

Methods

Acquisition of one dimensional signals using four circular coils spaced evenly across the FOV and positioned ¼ FOV above the object was simulated. Different coil sizes with radius ranging from 1/3 of the FOV to 1/8 of the FOV were simulated. aussian noise was added to the acquired data to achieve an SNR of 10. The acquired data were reconstructed with the GRAPPA technique using different numbers of neighboring lines. The coil weighted k-space SENSE matrix was also used to reconstruct the image from each coil. The band width of the SENSE matrix was calculated as the number of coefficients centered around the largest coefficient that was required to account for 95% of the total energy in each row of the matrix. The error for each individual coil image was measured as the sum of squares error at each signal point normalized by the sum of squares of signal

Results

An example of a k-space SENSE recon matrix for an acceleration factor of four is shown in Fig. 1. This matrix is composed of shifted blocks of four rows, which can be seen clearly by zooming in on a region near the diagonal of the matrix (Fig. 2). Fig. 3 shows a zoomed version of the GRAPPA matrix for the same dataset, which is also composed of shifted blocks of four rows. Notice the similar block circulant band structures of the SENSE and GRAPPA matrices. Fig. 6 shows the GRAPPA reconstruction error plotted against the width of the band of the SENSE matrix. From this plot, we see a trend that the wider the band of the SENSE matrix, the higher the error for GRAPPA reconstruction. GRAPPA can be viewed as an attempt to approximate the SENSE reconstruction process, so that the narrower the band of the SENSE matrix, the fewer GRAPPA coefficients are needed to achieve an accurate approximation.

In the critically determined case shown here, one line in each block of both the SENSE and GRAPPA matrices contains a single non-zero entry with value one. This pattern occurs in the SENSE matrix only in the critically determined case, indicating that acquired k-space lines are not modified by the reconstruction process. In the overdetermined case, data are combined from multiple coils to gain an SNR improvement, resulting in multiple non-zero entries in each row as shown in Fig. 4. In GRAPPA, this pattern will always occur because the reconstruction process is formulated so as to leave the acquired k-space lines unmodified (Fig. 5).

Discussion

In this work, we derived a k-space formulation of SENSE that reconstructs a unaliased image for each coil. The matrix used to achieve the recon is a block circulant matrix, and is approximately a band diagonal matrix for coils with smooth sensitivity profiles. The structure of this matrix provides justification for the structure of the convolution matrix used in GRAPPA. Thus, GRAPPA can be viewed as an approximation of SENSE in which the coefficients of a single truncated block of this matrix are estimated from autocalibration lines. A trend between the error of the GRAPPA reconstruction and width of the SENSE recon matrix's band was shown. This was expected because coil sensitivities that are dominated by low frequencies can be accurately approximated using a small number of Fourier coefficients. Therefore, when designing array coils, estimates of the coil sensitivity map can be used to estimate the width of the SENSE matrix's band, which can provide an indication of how well the GRAPPA reconstruction algorithm will perform for a given coil configuration.

References

[1] Griswold M, et al. MRM: 47:1202-1210, 2002. [2] Pruessmann KP, et al. MRM: 42:952-962, 1999. [3] Wang Y. MRM: 44:495-499, 2000. [4] Bini D, et al. Mathematics of Computation: 70:1169-1182, 2000.

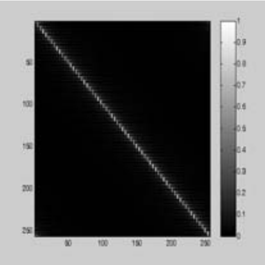

Fig. 1. SENSE recon matrix in k-space with reduction factor of 4

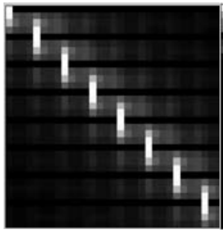

Fig. 2. Detail of the diagonal elements of the SENSE matrix in Figure

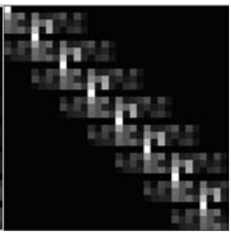

Fig. 3. GRAPPA recon matrix with reduction factor of 4

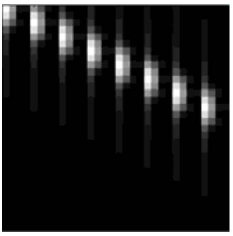

Fig 4. SENSE recon matrix in k-space with reduction factor of 2

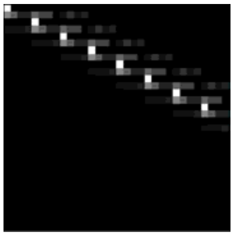

Fig 5. GRAPPA recon matrix with reduction factor of 2

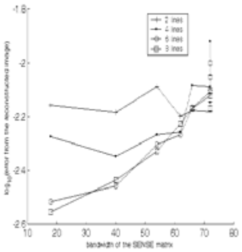

Fig 6. Error of the GRAPPA reconstruction