# Noise Reduction in Multiple Echo Data Sets using Singular Value Decomposition

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In MRI trade-offs usually have to be made between the signal to noise ratio (SNR), scan time and image quality, which can include the spatial resolution and contrast. As a general rule the SNR is increased by acquiring more data points and this can be achieved using any of four generic acquisition strategies in MRI; these are referred to as readouts, echos, repetition times (TR) and averages. Approximate time-scales associated with each acquisition are: $10^{-3}$ s, $10^{-2}$ s, $10^{0}$ s and $10^{2}$ s, respectively. Thus acquiring readout points is the most time efficient way to increase SNR while averaging is the least efficient. Unfortunately the readout duration cannot be increased indefinitely without sacrificing image quality and so less time-efficient data acquisition methods must be employed to gain higher SNR. All but one of the above methods are used routinely to increase SNR; the exception being echos, for which there is no widely adopted technique.

Several ideas have been proposed for combining images from different echo times (TE) but these tend to suffer from reduced spatial resolution and degraded contrast. One study proposed matched-filtering as a way of combining multiple echo data sets that gives the theoretically optimal SNR at full spatial resolution [1]. However the contrast in the final image is degraded since the highest signals contribute most to the final image. The approach of making weighted linear combinations of the data from different TEs is rather general and has been proposed independently with various SNR or contrast optimization criteria [2],[3]. The present study presents a method for jointly maximizing SNR and minimizing loss of contrast to produce an optimal estimate of the noise-free images.

## Theory

Noise reduction of time-series data is a well-studied area of signal processing. Optimal methods have been developed for this type of analysis – notably SVD, which decomposes the signal into a number of components ordered by their contribution to the variance in the signal. To determine the components for a multiple echo data set, the images of size $m \times n$ acquired at $p$ echo times is constructed into a matrix $\mathbf{H}$ with $mn$ rows and $p$ columns. The SVD of $\mathbf{H}$ returns three matrices

$$\mathbf{H} = \mathbf{U}\,\mathbf{S}\,\mathbf{V}^{\mathrm{T}} \qquad (1)$$

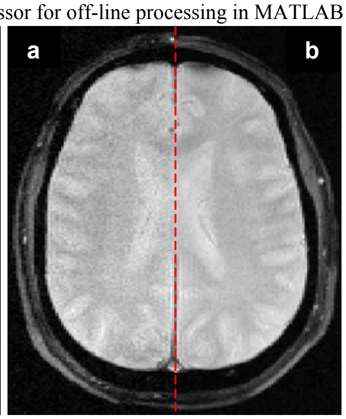
that are described in standard texts [4]–[5]. If $\mathbf{H}$ can be assumed to consist of the true signal $\overline{\mathbf{H}}$ contaminated with Gaussian random noise $\mathbf{N}$ then $\mathbf{H} = \overline{\mathbf{H}} + \mathbf{N}$ and the task of the processing is to find an approximation of $\overline{\mathbf{H}}$. It is well-known that $\overline{\mathbf{H}}$ can be approximated by zeroing all but the largest $k$ diagonal entries of $\mathbf{S}$. This approximation is optimal in the least squares sense, insofar as the Frobenius norm $\|\mathbf{H}_k - \mathbf{H}\|_{\mathbf{F}}$ is the minimum for all rank $k$ matrices and is suitable when the signal is a linear combination of exactly $k$ components corrupted with noise; for instance, if the signal comes from $k$ tissues each with its own distinct temporal variation . In practice, tissue variability and system imperfections create a mixture of variations therefore selecting a value for $k$ can be somewhat arbitrary. What may be preferable is a method that removes noise from $\mathbf{H}$ yet does not depend strongly on a particular parameter choice. Since $\mathbf{H}$ is sum of signal and noise, the singular values are

$$\sigma_j^2 = \overline{\sigma}_j^2 + \sigma_{noise}^2 \qquad j = 1,2,\ldots,p \qquad (2)$$

where $\overline{\sigma}_j$ are the singular values of $\overline{\mathbf{H}}$ and $\sigma_{noise}$ is one of the singular values of $\mathbf{N}$ [6]–[10]. The so-called "minimum variance" filter $f_j = 1 - \sigma_{noise}^2 / \sigma_j^2$ applied to the singular values of $\mathbf{H}$ produces the estimate $\mathbf{H}_{mv} = \mathbf{HT}$, where $\|\mathbf{HT} - \overline{\mathbf{H}}\|_{\mathbf{F}}$ is the minimum for all matrices $\mathbf{T}$. In other words, $\mathbf{H}_{mv}$ is the best approximation of $\overline{\mathbf{H}}$ that can be made by linearly combining the columns of $\mathbf{H}$. In the present application, the columns contain the image pixels at each TE so applying this filter results in the best approximation to the noise-free images that can be made from a linear combination of the noisy images. The parameter $\sigma_{noise}$ may be determined from the noise variance in the images using the relation $\sigma_{noise} = \sqrt{mn \times \text{variance}}$ or otherwise by assuming $\text{rank}(\overline{\mathbf{H}}) < p$, in which case $\sigma_{noise} = \sigma_p$.

## Method and Results

Data were acquired on a GE TwinSpeed 1.5T scanner using a spoiled gradient echo sequence. A birdcage head coil or 3-elements of the spine array were used. Data were transferred to a 3GHz P4 processor for off-line processing in MATLAB (The Mathworks, MA). Computation times were approximately 1–2 seconds per data set.



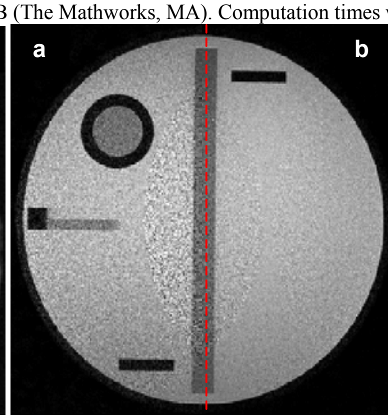




**Figure 1**
(a) Head image at TE 25.0 ms, matrix 256×144, α=35°, BW 545 Hz/pixel, TR 742 ms, TE = 5.2, 8.5, …, 31.6 ms, 9 echos in total. (b) the de-noised image.

**Figure 2**
(a) Phantom image at 4.0 ms with SENSE 3× acceleration [11], matrix 252×252, α=30°, BW 410 Hz/pixel, TR 100, TE = 4.0, 7.5, …, 28.6 ms, 8 echos in total. (b) the de-noised image.

**Algorithm**
An algorithm for the proposed method is given below. The input argument should contain a multiple echo data set and the output argument returns filtered images.

```
function output = SVD_filter(input)
    [m n p] = size(input);
    H = reshape(input,m.*n,p);
    [U S V] = svd(H,0);
    S = diag(S);
    f = 1 – (S(p)./S).^2;
    S = diag(f.*S);
    H = U*S*V';
    output = reshape(H,m,n,p);
```

## Discussion

Multiple echo combination techniques have been proposed previously [1]–[3] although have not become widely used in clinical practice. This may be due to the loss of contrast that can occur with these techniques and/or the unavailability of a fast and reliable method. The proposed technique puts forward a solution to both issues by simultaneously maximizing SNR and minimizing loss contrast (Figure 1) and making use of standard signal processing techniques. Strictly, the image noise must be Gaussian, whereas the noise in MRI depends on the method used for image reconstruction. The method can be modified to account for non-white noise [9] although the de-noising appears to be effective even when such steps are not taken (Figure 2).

Increasing the number of echos is a relatively time-efficient method of data acquisition compared to, say, signal averaging and may be viewed a way of extending the readout duration without incurring $T2^*$ blurring or chemical shift artifacts. Implementation of multiple echo sequences is relatively simple and, particularly with gradient echos, a large number of echos can be acquired in a short time by sampling on the gradient ramps and collecting data on both forward and reverse gradients. Chemical shift artifacts from using opposed readout gradient directions tend to get averaged away by the filtering, which may be an advantage in some cases.

## References

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