A theoretical analysis of errors in GRAPPA

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Introduction: GRAPPA [1] has been widely used as a prominent partially parallel imaging (PPI) technique. To understand GRAPPA better and hence further improve its performance, the error sources of GRAPPA are analyzed in this work. The relationship of the performance of GRAPPA to coil geometry, the choice of auto-calibrated signal (ACS) lines, blocks for reconstruction are all discussed based on the analysis of error sources. The discussion can be used to guide the design of coil and reconstruction algorithm. Linear coil distribution was used as an example to confirm some claims.

Theory :**Theoretical errors**: The missing *k*-space data can be approximated by using convolution in *k*-space. With given acquisition schemes (for example, acceleration factor, position of ACS lines) and sensitivity maps, the true convolution matrix *M* for channel *l* can be calculated by using the equation $M_l = C_m^l \left(C_a^H \psi^{-1} C_a\right)^{-1} C_a^H \psi^{-1}$, which was derived by Kholmovski *et al* [2]. C_m^l and C_a are the sub-blocks of the Fourier transform of the coil sensitivity matrix corresponding to the missing and acquired phase encoding lines, ψ is the noise correlation among channels, *H* is the conjugate operator. GRAPPA is a special case of convolution in *k*-space by using a truncated *M* with small convolution kernel. It can be seen from the definition of *M* that there are two kinds of theoretical errors for GRAPPA. One kind is caused by the truncation of the convolution kernel and is named truncation error; the other kind is caused by the inversion of matrix $A = C_a^H \psi^{-1} C_a$ and is named as inversion error. Based on the theory used by Xu *et al* [3], it can be deduced that the expectation of the L_2 norm of the reconstruction error of GRAPPA with the non-truncated *M* can be calculated by using Eq. 1 if the noise correlation matrix is identity. *tr(•)* is the trace operator for matrix. *E* is decided by the independence of the sensitivity maps and

$$E = \sum_l tr\left(C_m^l \left(\left(C_a^H C_a^H\right)^{-1}\right)\left(C_m^l\right)^H\right) \quad (1)$$

the acquisition scheme. The value of *E* can predict the inversion error caused by non-ideal coil geometry or acquisition scheme. **To reduce truncation error**, the coil needs to produce sensitivity maps that make *M* sparser, and hence more energy located at the neighbors used by GRAPPA. Obviously, smoother sensitivity maps can generate sparser *M*. If the sensitivity maps are not smooth, then *M* becomes denser and a larger convolution kernel is necessary to accuarately interpolate the missing data. When coil size is small or too close to the object, sensitivity maps will have hot spots hence more blocks are needed for accurate GRAPPA reconstruction. Therefore GRAPPA with sliding blocks generates better results for small size coil arrays. When there are several coils along Frequency encoding (FE) direction, then the sensitivity maps are no longer very smooth along FE direction, hence the element in *M* that corresponds to FE direction neighbors is no longer close to zero. For better reconstruction, a couple of blocks along FE directions are needed. This explains why GRASE [2], LIKE [3] generate better results for some coils. **To reduce inversion error**, the coil needs to produce sensitivity maps as independent as possible (this fights with the smooth requirement but the fighting can be solved by using larger convolution kernel) and the acquisition trajectory needs optimized. Hence the coil with gaps between loops has better performance for GRAPPA. The value of *E* can be used as a criterion for acqusition trajectory optimization (especially for vary density GRAPPA) to reduce inversion error. **Implentation errors**: With perfect sensitivity maps and full size convolution matrix, there are no truncation errors. However, the truncated convolution kernel is approximated by using ACS lines instead of perfect sensitivity maps, and hence it has implementation errror. The position, width of ACS lines and the number of unknowns (the size of the truncated convolution kernel) decides the kind of error. In most of the cases, the missing lines are located at high frequency region. When the true sensitivity maps are not smooth enough, the ACS lines located at low frequency can not accurately reflect the correlation of data in the high frequency region. Hence, the ACS lines without using the very central *k*-space lines can generate better results [4]. Even though increasing the size of convolution kernel reduces the truncation error, it also increases the number of unknowns when ACS lines are used to approximate the convolution kernel. With a fixed number of equations, more unknowns generate less accuracy of the approximation. Hence increasing the size of convolution kernel will increase the implementation error. Bigger convolution kernel does not always provide better reconstruction but always consumes longer time. The convolution matrix *M* can provide gudiance for the choice of convolution kernel to balance truncation error and implementation error.

Results: Some claims above have been proved in previous publications. In this work, a linearly distributed spine coil with 3 square loops was used for experiments to confirm two claims without existing experimental supports: “Smoother sensitivity maps need smaller convolution kernel” and “Increasing the size of convolution kernel does not always generate better results”. To change the smoothness of the sensitivity maps, the size of the loops were changed. One coil has loop size 10cm, the other one has loop size 13.8cm. Fig.1 shows the sensitivity maps of the two coils. Fig. 2 shows one row of the convolution matrix *M* for channel 1 when reduction factor is 3. Clearly, the coil with larger loop generates smoother sensitivity maps and hence sparser convolution matrix (Fig. 2b). To see if the actual performance for GRAPPA consistent with the analysis, full *k*-space phantom data were collected with the two kinds of coils on a 1.5 T GE system (TE 116, FOV 42cm, slice thickness 3mm). References were reconstructed by sum-of-squares with full k-space. PPI results were reconstructed by using GRAPPA with acceleration factor 3 and 24 ACS lines. Table 1 shows the result of the reconstruction by using these two coils. The first two columns show the relative error of the reconstructed images with different number of blocks. The third column shows the percentage of the weight of the neighbors used by 4 blocks GRAPPA in *M*. It can be seen that the convolution matrix *M* for smoother sensitivity maps is sparser and more energy is located in the neighbors used by GRAPPA and hence smoother sensitivity maps generated better results for GRAPPA (1st column); when more blocks were used, the performance of GRAPPA with small coil improved, but the larger coil (smoother sensitivity maps) generated worse result (2nd column). It demonstrates that “Smoother sensitivity maps need smaller convolution kernel” and “Increasing the size of convolution kernel does not always generate better results”. Another interesting observation is that the averge g-factor of the smaller coil (1.06) was better than that of the bigger coil (1.09). It demonstrates that the coil optimization based on g-factor does not always provide correct prediction for GRAPPA (based on the implementation).

Table 1. The results of GRAPPA

Coils	4 blocks	6 blocks	Weights
Smaller	7.24%	7.14%	8.03%
Larger	5.27%	5.53%	21.32%

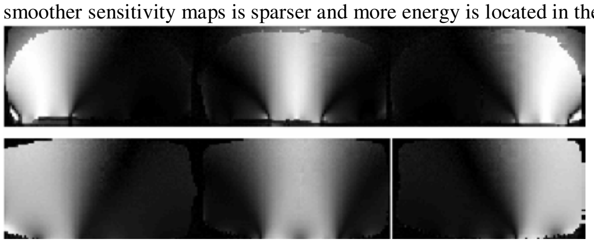

Fig. 1 Sensitivity maps of three square loops. Upper row is for smaller size coil; lower row is for the larger size coil

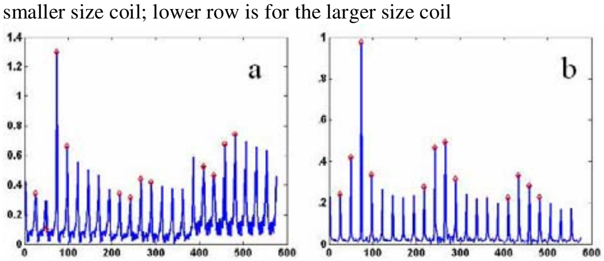

Fig. 2 One row of the convolution matrix. a) smaller size coil; b) larger size coil. The red diamonds show the weights used by 4 blocks GRAPPA. b) is much sparser than a)

Discussion: The performance of GRAPPA is closely related with the coil design, the choice of ACS lines and convolution kernel. For better reconstruction, the coil should generate more independent sensitivity map; for the implementation of GRAPPA with small size kernel, the coil should generate smooth sensitivity maps; the acquisition schemeshould be optimized to reduce the inversion error, the expectation *E* of inversion error can be used as the criterion for optimization. For different coils, the implementation of GRAPPA might be different (decided by the coil geometry) to get better reconstruction. Larger convolution kernel consumes longer reconstruction time but does not necessarily generate better results because the implementation error may surpass the truncation error. The truncation scheme, *i.e.* the choice of convolution kernel, can be guided by using the convolution kernel calculated by using sensitivity maps.

References: [1] Griswold, M.A., *et al.*, Magn Reson Med 47:1202-1210, 2002 [2] Kholmovski EG, *et al.* ISMRM 13, 2005, p 2257 [3] Huang F, *et al.*, ISMRM 12 2004 p 2139 [4] Park J, *et al.*, Magn Reson Med 2005;53:186-193.

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