Kriging and GRAPPA: A New Perspective on Parallel Imaging Reconstruction

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INTRODUCTION: Parallel MRI is widely used as a method for reduced acquisition. GRAPPA is a reconstruction method that recovers missing samples in the k-space directly using a weighted average estimation model [1]. In the field of geostatistics, a common method for spatial estimation is termed "kriging" [2]. This paper describes the application of kriging to the parallel imaging reconstruction problem. Furthermore, we show that GRAPPA is identical to kriging when considering Cartesian sampling.

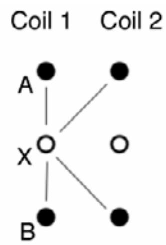

Fig. 1

THEORY: Assuming the simple case shown in Fig. 1, a local region of k-space is sampled at the points A and B using a two coil phased array receiver. The k-space signal at X on the *i*'th coil can be estimated by a simple weighted average at the two neighbors A and B and the multiple receiver signals, m_1 and m_2.

$$\hat{m}_i(\mathbf{k}_X) = w_{i_1} m_1(\mathbf{k}_A) + w_{i_2} m_1(\mathbf{k}_B) + w_{i_3} m_2(\mathbf{k}_A) + w_{i_4} m_2(\mathbf{k}_B) \quad (1)$$

Using GRAPPA, a matrix system corresponding to calibration data can be constructed to fit for the two sets of weights, $\mathbf{w}_i$.

$$\mathbf{G}\mathbf{w}_i = \mathbf{x}_i \quad (2)$$

Given M calibration points, $\mathbf{x}_i$, and N neighbors in the estimation model (Eq.1), the matrix **G** is M×N. The kriging approach instead tries to minimize the mean squared error between the estimated value and the true value at X.

$$\sigma^2_{\epsilon_i} = E\Big[\big(\hat{m}_i(\mathbf{k}_X) - m_i(\mathbf{k}_X)\big)\big(\hat{m}_i(\mathbf{k}_X) - m_i(\mathbf{k}_X)\big)^*\Big] \quad (3)$$

Under the assumption of ergodicity and second order stationarity the cross-covariance between k-space samples and channels is independent of the absolute k-space location and depends only on the relative vector distance **d**.

$$C_{ij}(\mathbf{d}_{\alpha\beta}) = E\Big[m_i(\mathbf{k}_\alpha) m^*_j(\mathbf{k}_\beta)\Big] \quad (4)$$

Setting the derivative of the error variance in Eq. 3 with respect to the real and imaginary parts of the weights to zero, taking advantage of the fact that the variance is real so the derivatives with respect to the real and imaginary variables can be recombined to complex form, and using the definition given in Eq. 4 lead to four equations (only one shown),

$$\frac{\delta\sigma^2_\epsilon}{\delta\mathcal{R}(w_{i_1})} + i\frac{\delta\sigma^2_\epsilon}{\delta\mathcal{I}(w_{i_1})} = w_{i_1} C_{11}(\mathbf{d}_{AA}) + w_{i_2} C_{11}(\mathbf{d}_{BA}) + w_{i_3} C_{21}(\mathbf{d}_{AA}) + w_{i_4} C_{21}(\mathbf{d}_{BA}) - C_{1i}(\mathbf{d}_{XA}) = 0 \quad (5)$$

which yields the kriging system.

$$\mathbf{K}\mathbf{w}_i = \mathbf{y}_i \quad (6)$$

The kriging matrix **K** is N×N and represents the cross-covariance between the k-space neighbors, A and B, while the right hand side contains the cross-covariance between the neighbors and the estimation point X. The kriging system can be shown to be equivalent to the system given by GRAPPA above if we multiply both sides of Eq. 2 by G^H.

$$\mathbf{K} = \mathbf{G}^H\mathbf{G} \qquad \mathbf{y}_i = \mathbf{G}^H\mathbf{x}_i \quad (7)$$

METHODS: Using a T1-weighted data set acquired with an eight-channel head coil on a human volunteer, rate 2 parallel images were generated using both GRAPPA and kriging. The estimation model in Eq. 1 included 4 points in the phase encoding direction and 7 points in the readout direction. Calibration lines were acquired to either compute the cross-covariances in Eq. 4, or fill the GRAPPA matrix in Eq. 2.

RESULTS: Fig. 2 displays rate 2 parallel reconstructions using both GRAPPA and kriging. The imaging results are identical, which is expected based on Eq. 7. Fig. 3 shows an example surface plot of the 2D cross-covariance measurement. Multiplication of the object by the coil sensitivities causes a blurring in the k-space. This blurring can be visualized by the cross-covariance, which is often called the covariogram.

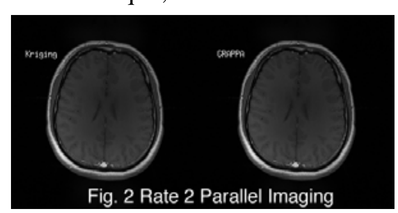

Fig. 2 Rate 2 Parallel Imaging

DISCUSSION: Kriging is a ubiquitous method in the field of geostatistics and mineral exploration. This paper demonstrates that the widely used GRAPPA method for parallel imaging can be formalized as a kriging problem. Kriging has been in development as a spatial estimation model for over 50 years with an established formalism, that implies a strong theoretical background for GRAPPA. Further investigation may show that kriging may be a more general method for parallel imaging reconstruction in the k-space.

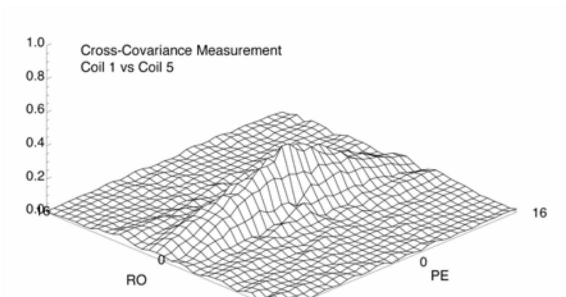

Fig. 3

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