Analytical Expressions for Noise Propagation in Diffusion Tensor Imaging

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Introduction

Noise in DWIs propagate into the calculated diffusion tensor and result in errors in the eigenvalues and eigenvectors. To quantify these errors a theoretical analysis is required. Although preliminary studies have been conducted to address this problem, no direct relationship between the SNR in DWIs and the errors in the calculated eigenvalues or eigenvectors has been established so far. For example, Poonawalla and Zhou[1] calculated the errors in the fractional anisotropy (FA) and relative anisotropy (RA) without dealing with the errors in the individual eigenvalues and eigenvectors. Anderson[2] used perturbation theory to investigate the effects of noise on eigenvalue and eigenvector estimations. However, this perturbation analysis is only valid for DWIs with moderate to high SNR. This restricts its practical application since DWIs typically suffer from low SNR. In this study, we derived analytical expressions based on error propagation theory[3] to determine the errors in the tensor eigenvalues and eigenvectors in terms of SNRs encountered in DWIs.

Methods

First, we establish a model for tensor error estimation, and secondly identify the eigenvalue and eigenvector errors separately. Since the determination of the diffusion tensor depends on the diffusion encoding scheme used, we adopt a widely used tensor encoding scheme[4] defined by Basser and Pierpaoli, in which the diffusion gradients are applied in the direction of [0,0,0], [1,0,1], [-1,0,1], [0,1,1], [0,1,-1], [1,1,0] and [-1,1,0] individually. The measurement intensity is represented respectively by S_0, S_1, S_2, S_3, S_4, S_5 and S_6. The tensor elements can be determined by[4] $D_{xz} = (-J_1+J_2)/(4b)$, $D_{yz} = (-J_3+J_4)/(4b)$, $D_{xy} = (-J_5+J_6)/(4b)$, $D_{zz} = (-J_1-J_2-J_3-J_4+J_5+J_6)/(4b)$, $D_{xx} = (-J_1-J_2+J_3+J_4-J_5-J_6)/(4b)$, $D_{yy} = (J_1+J_2-J_3-J_4-J_5-J_6)/(4b)$, with $J_1 = \ln(S_1/S_0)$, $J_2 = \ln(S_2/S_0)$, $J_3 = \ln(S_3/S_0)$, $J_4 = \ln(S_4/S_0)$, $J_5 = \ln(S_5/S_0)$, $J_6 = \ln(S_6/S_0)$, and $b = \gamma^2G^2\delta^2(\Delta-\delta/3)$. Applying error propagation theory[3] to these tensor calculation equations, the errors for each element of the tensor can be determined. Once the tensor errors have been quantified, the errors for the eigenvalues and eigenvectors can be identified further. The eigenvalues of the tensor can be determined by $\lambda_1 = I_1/3+(2/3)\mathrm{sqrt}(I_1^2-3I_2)\cos(\theta)$, $\lambda_2 = I_1/3+(2/3)\mathrm{sqrt}(I_1^2-3I_2)\cos(\theta-2\pi/3)$, $\lambda_3 = I_1/3+(2/3)\mathrm{sqrt}(I_1^2-3I_2)\cos(\theta+2\pi/3)$, with $I_1 = D_{xx}+D_{yy}+D_{zz}$, $I_2 = D_{xx}D_{yy}+D_{xx}D_{zz}+D_{yy}D_{zz}-(D_{xy}^2+D_{xz}^2+D_{yz}^2)$, $I_3 = D_{xx}D_{yy}D_{zz}+2D_{xy}D_{xz}D_{yz}-(D_{zz}D_{xy}^2+D_{yy}D_{xz}^2+D_{xx}D_{yz}^2)$, and $\theta = (1/3)\arccos[(1/2)(2I_1^3-9I_1I_2+27I_3)/(I_1^2-3I_2)^{3/2}]$. Conducting error propagation to the eigenvalue calculation equations, the errors on the eigenvalues can be quantified. Based on the result of eigenvalue errors, the errors of FA and RA can be identified further by using error propagation again. The principal eigenvector can be mathematically determined by[5] $e_{1x} = (T_1T_2)/\mathrm{sqrt}(T_1^2T_2^2+T_2^2T_3^2+T_1^2T_3^2)$, $e_{1y} = (T_2T_3)/\mathrm{sqrt}(T_1^2T_2^2+T_2^2T_3^2+T_1^2T_3^2)$, $e_{1z} = (T_1T_3)/\mathrm{sqrt}(T_1^2T_2^2+T_2^2T_3^2+T_1^2T_3^2)$, with $T_1 = D_{xy}D_{yz}-(D_{yy}-\lambda_1)D_{xz}$, $T_2 = D_{xz}D_{yz}-(D_{zz}-\lambda_1)D_{xy}$, $T_3 = D_{xz}D_{xy}-(D_{xx}-\lambda_1)D_{yz}$. Applying error propagation theory, the errors on the principal eigenvector can be determined.

To illustrate typical errors occurring in DTI data we consider a prolate tensor in the corpus callosum of a volunteer dataset. The signal intensity values for S_0 to S_6 were determined to be 200, 138, 95, 14, 79, 42, and 23, respectively. These values produce the following eigenvalues; $\lambda_1 = 0.0017\text{mm}^2/\text{s}$, $\lambda_2 = 0.0003\text{mm}^2/\text{s}$, and $\lambda_3 = 0.0001\text{mm}^2/\text{s}$ with $b = 1000\text{s/mm}^2$. The eigenvalues give FA = 0.871 and RA = 0.716. The SNR was defined in the range of 20 - 120 for the non-diffusion-weighted signal S_0. The noise level was thus within the range 1.7 to 10. We use the same noise level for all of S_0 - S_6 to determine the eigenvalue and eigenvector errors. As we can see, for the maximum noise level of 10, the SNR for the image with the greatest signal attenuation S_3 is reduced to 1.4.

Results

Figure 1 shows the eigenvalue and principal eigenvector errors as a function of the SNR of S_0. The errors for the eigenvalues and eigenvectors decrease as SNR increases. With SNR of 80, the relative error on λ_1 is 10%. However for λ_2 the error is 60% and for λ_3 the relative error is 100%. The large relative errors for λ_2 and λ_3 are due to their small values rendering their values imprecise. The principal eigenvector angular error at SNR of 80 is 6°. However, this error increases to 23° when SNR falls to 20. Figure 2 shows the FA and RA error as a function of SNR of S_0. The relative error for FA is clearly smaller than that of RA at any given SNR. This is shown in the righthand diagram of Figure 2 with SNR(FA)/SNR(RA) = 2.02, consistent with previous theoretical analysis[6].

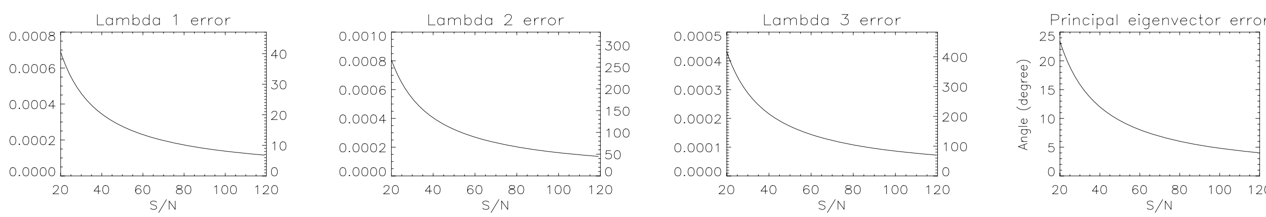

Figure 1. Plots of eigenvalue and principal eigenvector errors as a function of signal to noise ratio of S_0.

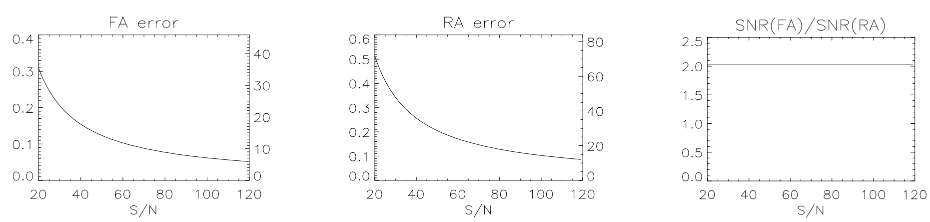

Figure 2. Plots of FA error, RA error and SNR(FA)/SNR(RA) as a function of signal to noise ratio of S_0.

Conclusion

We have established analytical expressions for the noise induced errors in the diffusion tensor elements, tensor derived metrics and the eigenvalues and eigenvectors. These expressions can be used to determine the errors for any given tensor geometry and for any SNR conditions. In particular these expressions can be used to evaluate errors in principal eigenvector orientation which has implications for diffusion tensor tractography. These expressions can also be used to aid DTI protocol design such as those aimed at detecting subtle changes in tensor parameters as a result of pathology.

Reference

[1]Poonawalla AH, Zhou XJ; J Magn Reson Imaging 2004; 19:489-498. [2]Anderson AW; Magn Reson Med 2001; 46:1174-1188. [3]Bevington P, Robinson DK; Data reduction and error analysis for the physical sciences. [4]Basser PJ, Pierpaoli C; Magn Reson Med 1998; 39:928-934. [5]Hasan, et al; J Magn Reson 2001; 152:41-47. [6]Hasan, et al; Magn Reson Med 2004; 51:413-417.