Efficient, Robust, Nonlinear, and Guaranteed Positive Definite Diffusion Tensor Estimation

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Problem: The relationship between diffusion-weighted images (DWIs) and the (apparent) diffusion tensor (DTI) is

(1) $$I^{(q)} = J\exp(-\mathbf{b}^{(q)} \bullet \mathbf{D}) + \text{noise}^{(q)}$$

where $I^{(q)}$ =image intensity for the q^{th} DWI (q=0,1,…), $\mathbf{b}^{(q)}$ =weighting matrix [e.g., $b_{ij}^{(q)} = \gamma^2 G_i^{(q)} G_j^{(q)} \delta^2 (\Delta - \delta/3)$], **D**=diffusion tensor (unknown), *J*="true" image intensity for **b=0** (unknown), and $\mathbf{b} \bullet \mathbf{D} \equiv \sum_{i,j} b_{ij} D_{ij}$. The simplest way to estimate the diffusion tensor from diffusion-weighted images is to take the logarithm of (1) and ignore the noise:

(2) $$-\log\left(I^{(q)} \big/ J\right) = \mathbf{b}^{(q)} \bullet \mathbf{D}$$

which is a set of linear relationships between the DWI data ($I^{(q)}$), scan parameters ($\mathbf{b}^{(q)}$), and the diffusion tensor (**D**) — assuming that *J* is known (usually $J = I^{(0)}$, acquired with $\mathbf{b}^{(0)} = \mathbf{0}$); **D** is solved for in (2) by linear least squares.

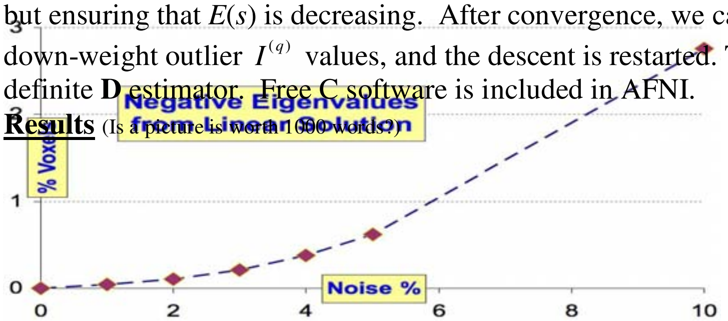
One difficulty with the log-linear approach is that the resulting **D** might not be positive definite (p.d.) [1]. A second difficulty is that the noise is not being treated properly [2] – linear least squares is appropriate when the noise is additive and each sample has the same variance; taking the logarithm violates both suppositions. When the eigenvalues of **D** are significantly disparate, these violations often result in poor estimates for **D**. We instead choose to fit **D** directly to (1), using a method sure to return a p.d matrix. Our method extends Tschumperle [3] to deal with (1) rather than (2).

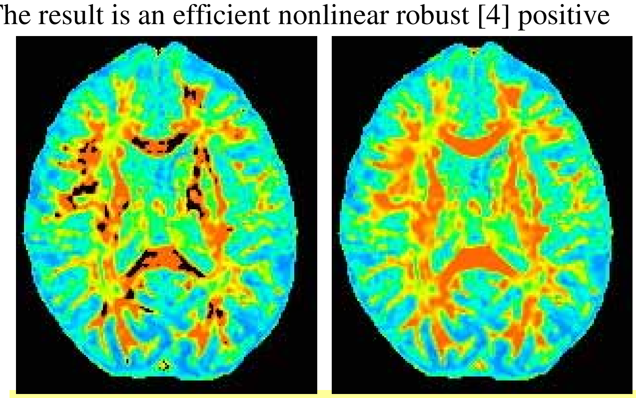
Solution: The goal is to find the value *J* and the symmetric p.d. matrix **D** that minimize the weighted error functional

(3) $$E(\mathbf{D}, J) = \tfrac{1}{2} \sum_q w_q \left[J\exp(-\mathbf{b} \bullet \mathbf{D}) - I^{(q)} \right]^2$$

As *J* appears quadratically, it can be estimated directly: $\hat{J}(\mathbf{D}) = \left[\sum_q w_q I^{(q)} \exp(-\mathbf{b}^{(q)} \bullet \mathbf{D})\right] \Big/ \left[\sum_q w_q \exp(-2\mathbf{b}^{(q)} \bullet \mathbf{D})\right]$. We use a modified gradient descent method to compute $\hat{\mathbf{D}}$ (minimizer of *E*). The gradient matrix of *E* wrt **D** is $\mathbf{F} = -\sum_q w_q \left[J\exp(-\mathbf{b}^{(q)} \bullet \mathbf{D}) - I^{(q)} \right] \mathbf{b}^{(q)}$. Pure gradient descent would solve the differential equation $\partial \mathbf{D}(s) \big/ \partial s = -\mathbf{F}(\mathbf{D})$, initialized with **D**(*s*=0) calculated via (2). However, this method of minimizing (3) often leads to an indefinite **D**. We use instead the fastest descent direction linear in **F** that *guarantees* **D** remains p.d., by solving $\partial \mathbf{D}(s) \big/ \partial s = -\left[\mathbf{F}(\mathbf{D})\mathbf{D}^2 + \mathbf{D}^2 \mathbf{F}(\mathbf{D}) \right]$; along this curve, $\partial E \big/ \partial s = -2 \|\mathbf{FD}\|^2$. The descent curve is computed using a Padé approximant method consistent with this equation, which also ensures **D** remains p.d. even for finite stepsizes: define $\mathbf{H}_{\pm}(\varepsilon) = \mathbf{I} \pm \tfrac{1}{2} \varepsilon \mathbf{FD}$, and then $\mathbf{D}(s+\varepsilon) = \mathbf{H}_{-}(\varepsilon) \mathbf{H}_{+}(\varepsilon)^{-1} \mathbf{D}(s) \mathbf{H}_{+}(\varepsilon)^{-1} \mathbf{H}_{-}(\varepsilon)$. The stepsize $\boldsymbol{\varepsilon}$ is chosen as large as possible, but ensuring that *E*(*s*) is decreasing. After convergence, we calculate the residuals; the initial weights are modified to down-weight outlier $I^{(q)}$ values, and the descent is restarted. The result is an efficient nonlinear robust [4] positive definite **D** estimator. Free C software is included in AFNI.

Results (Is a picture is worth 1000 words?)





Fractional anisotropy with linear (LEFT) and nonlinear (RIGHT) methods; voxels with negative eigenvalues are black.

References:

[1] Skare et al, *Mag.Reson.Imaging*, **18**:659-669 (2000). **[2]** Jones & Basser, *Magn.Reson.Med.*, **52**:979-993 (2004).
[3] Tschumperle & Deriche, *Lect.Notes Comp.Sci.* **2809**:530-541 (2003). **[4]** Mangin et al., *Med.Imag.Analysis* **6**:191-198 (2003)