Optimal Sampling of k-Space with Cartesian Grids for Parallel MR Imaging

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INTRODUCTION

Image quality in parallel MR imaging depends on three main factors: (1) coils' sensitivity functions of the receiver system, (2) *k*-space sampling locations of the measured data, and (3) reconstruction methods. The problem of how to optimally sample *k*-space has received much less attention than the other two so far because of the complexity of the problem. Early parallel imaging methods, such as SMASH [1] and Cartesian SENSE [2], sample *k*-space using uniform Cartesian grids. It was soon recognized that non-uniform sampling can yield much better reconstructed images. SPACE-RIP [3], for example, can produce higher-quality images than the conventional SENSE by using heuristic *k*-space sampling patterns. A more rigorous approach was introduced recently in [4], where the desired image function is assumed to be a stationary Gaussian random process and the minimum mean-squared-error (MMSE) criterion is used to select the "optimal" *k*-space locations for collecting the phase-encoded data. This paper addresses the *k*-space sampling problem rigorously from a new perspective. Specifically, a signal-independent criterion was derived to guide the selection of optimal *k*-space locations for data collection, and a novel fast algorithm, based on sequential backward selection (SBS) [5], was proposed to solve the underlying optimization problem efficiently. Experimental results have demonstrated a significant improvement of the proposed method over SENSE and SPACE-RIP in terms of signal-to-noise-ratio (SNR) and aliasing artifacts. Theoretical analysis has also been carried out, which shows that the sampling patterns selected by the proposed algorithm are robust to small variations on the coils' sensitivity functions.

PROPOSED METHOD

Optimal sampling with Cartesian grids (OSCAR) is studied in this paper. We assume a phased array of L coils and an $N \times N$ image. The goal is to choose M phase encoding lines out of the N uniformly spaced ones (each with N samples along the read-out direction) that give the minimum sum-of-squared error (SSE).

A. Optimality Criterion

We consider the same imaging equation, $\tilde{\mathbf{S}}\boldsymbol{\rho} = \tilde{\mathbf{d}}$, as in [3], where the ($N^2 \times 1$) vector $\boldsymbol{\rho}$ is the vectorized image, $\tilde{\mathbf{S}}$ is obtained by choosing LMN rows of the $LN^2 \times N^2$ sensitivity profile matrix $\mathbf{S}$. Similarly $\tilde{\mathbf{d}}$ is obtained from $\mathbf{d}$ (vectorized measurements) by choosing its corresponding elements. Assuming Gaussian noise, the minimum variance reconstruction is given by $\boldsymbol{\rho}_{\mathrm{MV}} = (\tilde{\mathbf{S}}^{\mathrm{H}}\boldsymbol{\Psi}^{-1}\tilde{\mathbf{S}})^{-1}\tilde{\mathbf{S}}^{\mathrm{H}}\boldsymbol{\Psi}^{-1}\tilde{\mathbf{d}}$, and the SSE of the reconstructed image is

$$\mathrm{E}\left(\left\|\boldsymbol{\rho} - \boldsymbol{\rho}_{\mathrm{MV}}\right\|^2\right) = \mathrm{trace}\left((\tilde{\mathbf{S}}^{\mathrm{H}}\boldsymbol{\Psi}^{-1}\tilde{\mathbf{S}})^{-1}\right), \tag{1}$$

where $\boldsymbol{\Psi}$ is the noise correlation matrix. Thus, the optimal selection procedure boils down to the selection of the $LMN \times N^2$ sub-matrix $\tilde{\mathbf{S}}$ that gives the smallest value of (1). For simplicity, we consider the case where $\boldsymbol{\Psi}$ is identity matrix. The general case can be dealt with by using a Cholesky factorization [6] of $\boldsymbol{\Psi}$ followed by a change of variables.

B. Fast Algorithm

Determination of the M best phase encoding locations by exhaustive search is a combinatorial optimization problem whose complexity is prohibitive for practical application. We overcome this problem by generalizing the SBS algorithm [3] which provides a sub-optimal, but fast solution. Specifically, we start from the full encoding matrix $\mathbf{S}$ and sequentially eliminate blocks of L rows (each block corresponds to N samples of one phase encode line). At each step, we eliminate the blocks that give the least increment in the cost. Instead of directly calculating Eq. (1), we recursively update its value according to Sherman-Morrison formula [6]. We use a series of speed-up techniques such as storing $(\tilde{\mathbf{S}}^{\mathrm{H}}\tilde{\mathbf{S}})^{-1}\mathbf{S}^{\mathrm{H}}$ and utilizing the separability of 2D FFT. The whole process continues until M phase encodings are left.

The algorithm needs $O(N^4)$ multiplies, as compared to $O(N^6)$ multiplies if (1) were evaluated directly. The time taken for finding the optimal phase encoding locations for a 128×128 image is about 5 minutes on a 2.66GHz PC.

C. Perturbation Analysis

In practical applications, the sensitivity profiles are estimated; the estimation errors $\Delta\mathbf{S}^*$ can lead to degradation of image quality. Denoting $f(\mathbf{S}^*)$ as SSE in the reconstruction corresponding to the optimal $\mathbf{S}^*$, we obtain

$$\left| f(\mathbf{S}^* + \Delta\mathbf{S}^*) - f(\mathbf{S}^*)\right| / \left| f(\mathbf{S}^*)\right| \le 2\kappa(\mathbf{S}^*)\left\|\Delta\mathbf{S}^*\right\|_{\mathrm{F}} / \left\|\mathbf{S}^*\right\|_{\mathrm{F}} \tag{2}$$

where $\kappa(\cdot)$ is the condition number and $\|\cdot\|_{\mathrm{F}}$ is the Frobenius norm. The above formula indicates that the drop-off in the image quality is insensitive to small perturbations of $\mathbf{S}^*$ when the condition number is small, which is the case when the number of phase encoding lines is not close to the limit.

RESULTS

A set of representative results is shown in Fig. 1, where a T_1-weighted brain data was acquired using a four-coil array. Data from 54 phase encoding lines (reduction factor R = 2.37) was used for SENSE, SPACE-RIP and OSCAR. As shown in Fig. 1, OSCAR achieves the smallest SSE with minimum image artifacts. Much smaller geometry factors than SENSE and much smaller area of high geometry factors than SPACE-RIP are observed.

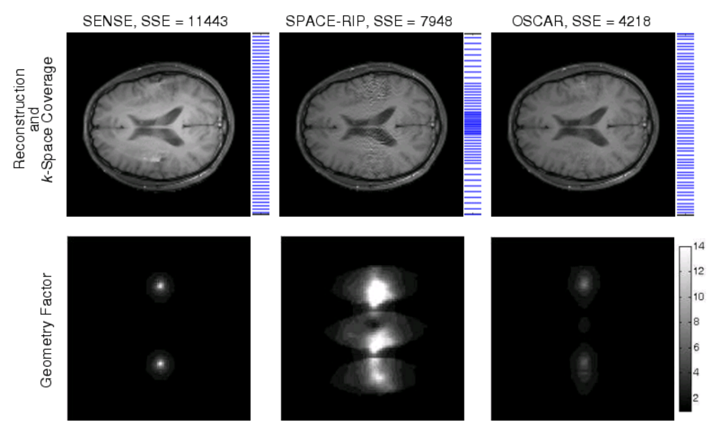

Fig. 1. Reconstructions using uniform (SENSE), variable-density (SPACE-RIP) and optimal (OSCAR) samplings. The *k*-space coverage is depicted besides each reconstruction. Geometry factors are also plotted under corresponding reconstruction.

CONCLUSION

A new method is proposed to optimize *k*-space locations for data collection in parallel imaging. Experimental results show that the proposed method improves SNR and reduces aliasing artifacts of the reconstructed images as compared to SENSE and SPACE-RIP. The proposed method should prove useful for a range of practical applications where it is desirable to use optimal sampling of *k*-space to further improve the imaging speed and quality of parallel imaging.

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