# Can Maximum Likelihood Estimation Outperform Matrix Inversion in Parallel Image Reconstruction?

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**Introduction** A close correspondence has recently been reported between parallel magnetic resonance imaging and multiple-input multiple-output (MIMO) wireless communications (1). Both fields utilize measured "sensitivity" information in multiple detectors to unfold aliased data. However, in MIMO, maximum likelihood estimation (MLE) has been shown to outperform matrix inversion techniques for signal decoding (2). One might speculate, therefore, that MLE might also yield improved performance for parallel MRI, in particular if the reconstructed signal data assume finite bit precision. This study addresses this question by first outlining a general theory of MLE and also the special conditions under which MLE and encoding matrix inversion become mathematically equivalent. A practical adaptation of MLE for parallel image reconstruction with finite bit precision is then described. Lastly, the results of numerical simulations comparing MLE and matrix inversion are presented and discussed.

**Theory** An MLE algorithm selects a solution vector $\hat{\mathbf{x}}$ (out of an exhaustive set of candidate **x**'s) which maximizes the conditional probability of the observation vector **y** given **x**.

$$\hat{\mathbf{x}} = \underset{\mathbf{x}\,\in\,\text{all possible x's}}{\arg\max}\; p_{(\mathbf{y}|\mathbf{x})}\left(\mathbf{y}|\mathbf{x}\right);\quad \text{where } \mathbf{y} = f\left(\mathbf{x}\right) + \mathbf{n}; \qquad [1]$$

In typical cases for both parallel MRI and MIMO, $f(\cdot)$ is linear and invertible such that part of Eq. [1] can be written in matrix form,

$$\mathbf{y} = \mathbf{Fx} + \mathbf{n};\quad \mathbf{F}^{-1}\mathbf{F} = \mathbf{I}_{(identity)}; \qquad [2]$$

and **n**'s are noise of normal distribution with covariance matrix $\mathbf{\Lambda}$. However, parallel imaging differs from MIMO in that, in parallel imaging, **x** assumes continuous values representing the underlying magnetization density, whereas in MIMO, **x** takes discrete values from a finite alphabet. For the continuous case represented by the parallel imaging reconstruction problem, the MLE solution mathematically coincides with the least-squares "pseudoinverse" solution given by linear algebra (3),

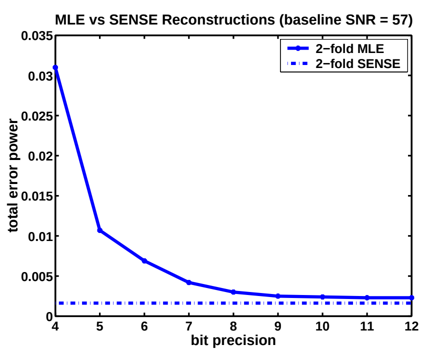
$$\hat{\mathbf{x}} = \mathbf{F}^{-1}\mathbf{y},\quad \text{where } \mathbf{F}^{-1} = \left(\mathbf{F}^{\mathrm{T}}\mathbf{\Lambda}^{-1}\mathbf{F}\right)^{-1}\mathbf{F}^{\mathrm{T}}\mathbf{\Lambda}^{-1} \qquad [3]$$

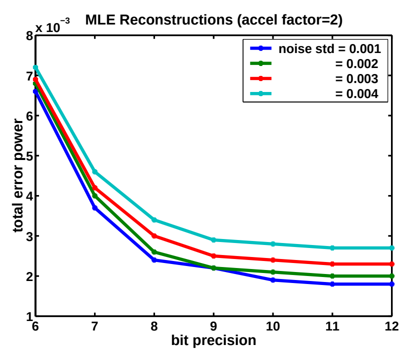
On the other hand, the discreteness of **x** in MIMO allows "non-linear" decoding using MLE, and this stochastic approach outperforms the linear algebraic least-squares approach, which is sometimes referred to in this context as "zero-forcing"(1). This study investigates whether the MLE approach can also outperform the matrix least-squares approach in parallel MRI when the estimated $\hat{\mathbf{x}}$ assumes finite bit precision.

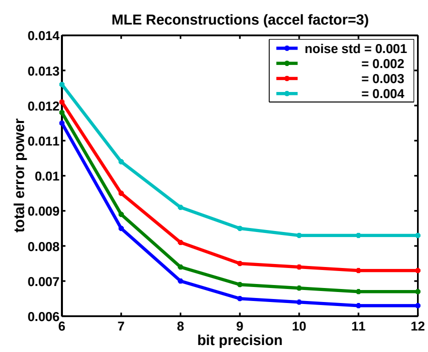
**Method** The Viterbi algorithm (4), a computationally efficient MLE decoder for convolutional codes in wireless system, was adapted to perform MLE for parallel imaging. The encoding methods for convolutional codes and MR signal data are schematically analogous since *k*-space data can be expressed in terms of the convolution between the Fourier transformations of the spin density and of the coil sensitivities. Low-resolution coil sensitivities (central 8 Fourier coefficients) were used in order to reduce the complexity of the Viterbi algorithm, which scales exponentially with the length of the convolutional kernel. By initially quantizing $\hat{\mathbf{x}}$ to a limited bit precision, a coarse MLE solution can be obtained after the first iteration. Successive iterations add bit precision to the MLE solution by refining the quantization levels. Each iteration adds ½ bit of resolution to the MLE solution. A total of 24 iterations were required to obtain the 12-bit precision that is typical in MR images.

Signal data were simulated using a coil array of 8 rectangular elements (400x60mm each), covering a total area of 400x400mm (also the field of view of the image). Various levels of Gaussian noise were added. Parallel image reconstructions were performed from datasets with various acceleration factors, using both the MLE algorithm described above and the standard SENSE least-squares pseudo-inverse reconstruction (5). Total error power was plotted as a function of bit precision.

**Results**




**Discussion** This work tests the potential benefit of MLE for parallel image reconstructions of finite bit precision. Preliminary results suggest that assuming a solution of moderate but finite bit precision gives *no* apparent advantage over the linear least-squares approach. In Fig 1, numerical simulations confirm theoretical expectations that, in the limit of infinite bit resolution, the two approaches converge. The error plots in Fig. 2 and 3 show that total error power does cease to decrease at some bit level, suggesting that the baseline SNR ultimately dictates the realizable bit precision of an image. One might imagine only very specialized circumstances with limited precision requirements for which the MLE algorithm might provide a specific SNR advantage. That said, the stochastic approach in MLE does offer new generalities in accommodating different noise behaviors (e.g. non-Gaussian), *a priori* information, and non-linear solutions for future developments of parallel imaging.

**Reference 1.** Bolcskei H, 2nd Parallel MRI Workshop, 2004:91. **2.** Bolcskei H, *The Communication Handbook*, 2nd ed, CRC Press, p90.1-14. **3.** Saridis, GN, *Stochastic Processes, Estimation, and Control*, John Wiley & Sons:p62. **4.** Proakis, JG, *Digital Communications* 3rd ed. McGraw-Hill, p483-6. **5.** Pruessmann, KP, MRM 1999;42(5):952-62.