Fast, Optimal Weighting for Image Reconstruction from Arbitrary k-Space Trajectories

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Introduction

In MRI, one seeks to produce an accurate image of the effective spin density, ρ(**r**), from a set of measurements $\{s_q\}$ acquired in the Fourier domain. Because the signal equation (1a) is a continuous-to-discrete mapping, image reconstruction is inherently ill-posed, i.e., infinitely many functions give rise to the same data. Complicating matters, fast imaging sequences use trajectories that result in a non-uniform sampling of k-space. One is therefore forced to choose a reconstruction procedure, either a direct method, e.g. the pseudo-inverse [1] or "convolution-gridding" [2], or an iterative one [3]. We describe a unified analytical framework for understanding all of these reconstruction schemes. We derive a simple analytical formula for optimal density compensation weights and show that the computation can be carried out in O(NlogN) time. The combination of non-uniform FFT and these optimal weights results in an extremely fast and accurate method to reconstruct MR images from arbitrarily sampled k-space trajectories.

Background

$$s(\mathbf{k}_q) = \int_V \rho(\mathbf{r}) e^{2\pi\imath \mathbf{k}_q\cdot\mathbf{r}} d\mathbf{r} \qquad (1a)$$

$$\rho(\mathbf{r}) = \int s(\mathbf{k}) e^{-2\pi\imath \mathbf{k}\cdot\mathbf{r}} d\mathbf{k} \qquad (1b)$$

$$\rho(\mathbf{r}) \approx \sum_q w_q s_q e^{-2\pi\imath \mathbf{k}_q\cdot\mathbf{r}} \qquad (2)$$

Fourier Reconstruction: The signal equation (1a) expresses s(**k**) as the Fourier transform of ρ(**r**) and satisfies the inverse formula (1b). With only a finite set of measurements, one can seek an approximation to the integral transform (1b) as a sum (2) with the factors w_q serving as quadrature weights. There are thus two tasks: 1) selecting appropriate weights and 2) implementing (2) efficiently. During the past several years the latter problem has been solved. Sums of the form (2) can be computed in O(N log N) time with complete control of precision using the non-uniform FFT (NUFFT) [4,5,6]. It is worth emphasizing that once the decision has been made to use a reconstruction of the form (2), the issues of fast computation and weight selection are completely independent of one another.

$$\mathbf{s} = F\rho(\mathbf{r}) \qquad (3a)$$

$$\hat\rho(\mathbf{r}) = F^+\mathbf{s} = F^*Q^+\mathbf{s} \qquad (3b)$$

$$Q_{mn} \equiv (FF^*)_{mn} = sinc(k_m - k_n) \qquad (3c)$$

$$\hat\rho(\mathbf{r}) = \sum_q\Big(\sum_r Q^+_{qr}s_r\Big)e^{-\imath\mathbf{k}_q\mathbf{r}} \qquad (3d)$$

Pseudo-Inverse, Least Squares Reconstruction: Another possible image reconstruction scheme is the pseudo-inverse [7]. The procedure as outlined by [1] is reproduced in (3a-d). The forward operator is F, and its pseudo-inverse is $F^*(FF^*)^+$. The singular value decomposition (SVD) can be used to invert the NxN matrix $Q=FF^*$, an expensive procedure requiring $O(N^3)$ operations, feasible for moderate sized 2-D acquisitions (N=64x64), but impractical at higher resolution or in 3-D. Computational complexity aside, in the absence of *a priori* information, the pseudo-inverse reconstruction stands out as a generally agreed upon standard for comparison.

$$\hat\rho = F^*Q^+s = F^*Q^+F\rho \approx \rho \qquad (4a)$$

$$min_W \|\mathbb{I} - F^*WF\|_F \qquad (4b)$$

$$M\mathbf{w} = \mathbf{1} \qquad (4c)$$

$$M_{mn} = sinc^2(k_m - k_n) \qquad (4d)$$

Density Compensation Weights: Several weighting schemes have been considered in the literature based on 1) analytically known trajectories and their Jacobian mapping [8,9], 2) area weighting using Voronoi tesselation [10], 3) convolution of delta-functions at the nodes $\{k_q\}$ with a kernel [2];, and 4) iterative methods [11,12]. While all of these approaches are reasonable, they don't satisfy any clear optimality condition.

Optimal Weights: Recently, Sedarat and Nishimura [13] described an optimal weighting scheme. They astutely observed that gridding reconstruction (2) is an approximation to pseudo-inverse reconstruction (3), and that the density compensation weights in (2) are the equivalent of a diagonal approximation to Q^+ in (3). They show that the optimal weights can be obtained by minimizing the error in (4a), solving the linear system (4c). As they note, (4c) is ill-conditioned and requires the SVD for solution, again requiring $O(N^3)$ work.

$$\hat s = FF^*Q^+s \approx s \qquad (5a)$$

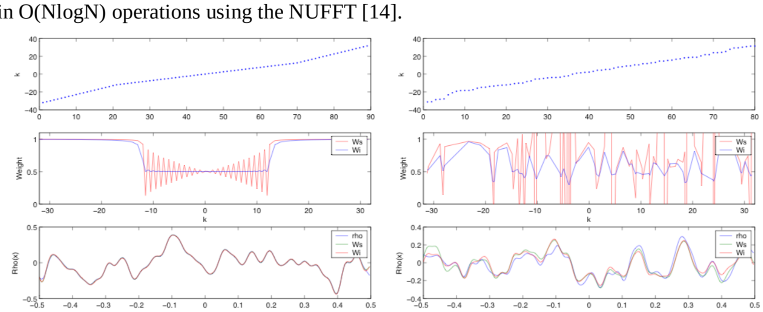
$$min_W \|\mathbb{I} - QW\|_F \qquad (5b)$$

$$M\mathbf{1} = 1/\mathbf{w} \qquad (5c)$$

$$w_k = \frac{1}{\sum_j M_{jk}} \qquad (5d)$$

Results and Discussion

Fast Analytic Optimal Weights: **A.** We propose a slight change to the formalism, which has surprising impact. Instead of minimizing (4a) the error in the image domain, we consider the dual approach (5a) and minimize the error in the signal domain. A similar calculation to the one performed in [13] results in the relationship (5c) and the analytic formula (5d). **B.** Because the matrix M is of the form of a discrete convolution, the formula (5d) can be computed in O(NlogN) operations using the NUFFT [14].



Simulations: We present results from simple 1-D Matlab simulations to demonstrate the accuracy of the reconstruction obtained using the optimal weights (5d). A random bandpass-limited object was used to generate the signal and reconstruction was carried out according to (2) with weights satisfying (4c) and (5c). *Left Panel*: trajectory with double the sampling density near the center of k-space. *Right Panel*: trajectory with 80 random points. For each panel, the sampling scheme is at the top, the weights are in the middle (Red for (4c), Blue for (5d)), and the object and reconstructions are at the bottom (Blue for original, Green for (4c), Red of (5d)). In both cases the reconstruction is extremely accurate, although, as expected, the random trajectory produces somewhat worse reconstructions of the true object because of gaps in the sampling. Note that for these redundant samplings, the weights (4c) from [13] have an oscillatory behavior, which is undesirable if they are to be interpreted as quadrature weights. Regularization in solving (4c) controls the oscillations somewhat, but a clear rule for this is not presently available.

Conclusion

We have developed optimal density compensation weights for arbitrary 1, 2, and 3-D k-space trajectories computable on the fly, in O(NlogN) time. We believe that these weights, in conjunction with the NUFFT, will result in extremely fast and accurate MR image reconstruction from arbitrarily sampled k-space trajectories.

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