Fast SENSE Reconstruction Using Linear System Transfer Function

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INTRODUCTION: Reconstructing images from sensitivity encoded (SENSE) data acquired on arbitrary k-space trajectories is computationally demanding (1,2). Iterative reconstructions using the conjugate gradient (CG) method have been shown to be effective for various trajectory types (2). With the CG method, a matrix-vector product has to be calculated during each iteration. To improve the reconstruction efficiency, this multiplication is replaced with two gridding steps (2). However, gridding can still be computationally intensive for high resolution applications and for a large number of receiver coils. Even with preconditioning, to reconstruct an image from these data, it will take minutes to converge for the CG algorithm based on forward and backward gridding. Here, we present a fast SENSE reconstruction algorithm by utilizing the property of system linearity. We show that a system transfer function can be derived and used to realize the matrix-vector multiplication, which dramatically improves the reconstruction efficiency.

METHOD: In each iteration step of the SENSE reconstruction, the product of $(\mathbf{E}^H\mathbf{DE})\mathbf{x}$ has to be computed (2). Here $\mathbf{E}$ is an encoding matrix; $\mathbf{H}$ represents Hermitian operation; $\mathbf{D}$ is the k-space density correction function; $\mathbf{x}$ is an intermediate image weighted by coil sensitivities. Specifically, one needs to compute,

$$\mathbf{f}(\mathbf{r}_\rho) = (\mathbf{E}^H\mathbf{DE})\mathbf{x} = \sum_k \sum_{\rho'} D(\mathbf{k}_k)\exp(i\mathbf{k}_k(\mathbf{r}_\rho - \mathbf{r}_{\rho'}))\,\mathbf{x}(\mathbf{r}_{\rho'}) \quad [1]$$

Here $\mathbf{r}_\rho$ is a position vector in the image domain and $\mathbf{k}_k$ is a sampling position in the **k**-space, which may not be located on a regular Cartesian grid. Eq.[1] describes a 2D linear shift invariant system (LSI) with $\mathbf{x}(\mathbf{r}_\rho)$ as input and $\mathbf{f}(\mathbf{r}_\rho)$ as output. Its transfer function can be found in the spatial frequency domain. Since $\mathbf{x}(\mathbf{r}_\rho)$ and $\mathbf{f}(\mathbf{r}_\rho)$ are sampled on regular Cartesian grids, the discrete Fourier transform (DFT) can be applied to find their spectrums on a Cartesian grid **k**:

$$\mathbf{F}(\mathbf{k}) = \sum_\rho \sum_k \sum_{\rho'} \exp(-i\mathbf{k}\mathbf{r}_\rho)\,D(\mathbf{k}_k)\exp(i\mathbf{k}_k(\mathbf{r}_\rho - \mathbf{r}_{\rho'}))\,\mathbf{x}(\mathbf{r}_{\rho'}) \quad [2]$$

Let $\mathbf{r}_q = \mathbf{r}_\rho - \mathbf{r}_{\rho'}$, and multiply Eq.[2] with $\exp(-i\mathbf{k}\mathbf{r}_{\rho'})\exp(i\mathbf{k}\mathbf{r}_{\rho'})$. Eq. [2] can be rewritten as:

$$\mathbf{F}(\mathbf{k}) = \left[\sum_q \exp(-i\mathbf{k}\mathbf{r}_q)\sum_k D(\mathbf{k}_k)\exp(i\mathbf{k}_k\mathbf{r}_q)\right]\sum_{\rho'}\exp(-i\mathbf{k}\mathbf{r}_{\rho'})\mathbf{x}(\mathbf{r}_{\rho'}) = \left[\sum_q \exp(-i\mathbf{k}\mathbf{r}_q)\sum_k D(\mathbf{k}_k)\exp(i\mathbf{k}_k\mathbf{r}_q)\right]\mathbf{X}(\mathbf{k}) \quad [3]$$

Therefore, the system transfer function can be expressed as:

$$\mathbf{H}(\mathbf{k}) = \sum_q \exp(-i\mathbf{k}\mathbf{r}_q)\sum_k D(\mathbf{k}_k)\exp(i\mathbf{k}_k\mathbf{r}_q) = \sum_k D(\mathbf{k}_k)\sum_q \exp(-i(\mathbf{k}-\mathbf{k}_k)\mathbf{r}_q) = \sum_k D(\mathbf{k}_k)\,G(\mathbf{k}-\mathbf{k}_k) \quad [4]$$

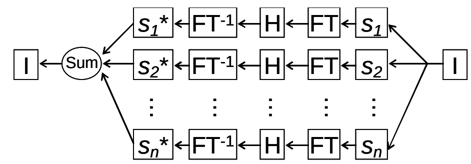


Fig 1 – Flow chart of the transfer function based SENSE. This part is to be incorporated into the conjugate gradient iteration. s_n is the coil sensitivity of the n-th coil. **H** represents transfer function multiplication in k-space. **I** is intensity correction.

Since $\mathbf{r}_q$ is on Cartesian grids, $G(\mathbf{k}\text{-}\mathbf{k}_k)$ can be computed analytically. When $\mathbf{k}_k$ fully samples **k**-space, $G(\mathbf{k}\text{-}\mathbf{k}_k)$ is the Fourier transform of a matrix with all ones sampled at locations $\mathbf{k}\text{-}\mathbf{k}_k$. It can be approximated by a delta function, $\delta(\mathbf{k}\text{-}\mathbf{k}_k)$. For this special case, we find that $\mathbf{H}(\mathbf{k}) = D(\mathbf{k})$, where $D(\mathbf{k})$ is the **k**-space density correction function evaluated on a Cartesian grid. For general cases, $\mathbf{H}(\mathbf{k})$ can be computed using Eq.[4]; and $\mathbf{H}(\mathbf{k})$ only needs to be computed once along with the initial image.

Overall, the product of $(\mathbf{E}^H\mathbf{DE})\mathbf{x}$ can be computed as $\mathrm{FFT}^{-1}(\mathbf{H}(\mathbf{k})\mathbf{X}(\mathbf{k}))$, where $\mathbf{H}(\mathbf{k})\mathbf{X}(\mathbf{k})$ represents an element-by-element multiplication and FFT^{-1} is the inverse fast Fourier transform (Fig 1). Compared to the gridding steps, the transfer function approach reduces the number of multiplications from $2(2N)^2w^2$ to N^2. Here N is the image size and w is the size of the convolution kernel chosen in the gridding procedure. The resulting speed-up factor is thus around $8w^2$.

The transfer-function SENSE algorithm was tested both on simulated and in vivo data. In both cases, an 8-channel receiving coil was used. Two types of trajectories were designed: a Cartesian grid and a variable-density (VD) spiral (3). The spiral trajectory contained 20 interleaves; and each interleaf had 4484 sampling points. The trajectory was designed with a field of view (FOV) of 22 cm for an acquisition matrix of 256x256. Simulated k-space data were generated using a brute-force DFT of a fast-spin-echo (FSE) phantom image. In vivo brain data were acquired on a healthy volunteer with a GE Signa 1.5T whole-body system using a spin echo (SE) sequence (TR = 2.5s and TE = 68ms). Coil sensitivity maps were measured in a separate calibration scan using both body coil and 8-channel receiver coil.

RESULTS: Figure 2 shows images reconstructed from Cartesian data. Reduction factor ranges from one to four. The reduction factor is shown in the upper right corner of each image. Fig 2a shows simulated phantom images. Fig 2b shows in vivo results. Figure 3 shows images reconstructed from VD spiral data. Similarly, images in Fig 3a are reconstructed from simulated data; and images in Fig 3b are reconstructed from in vivo data.

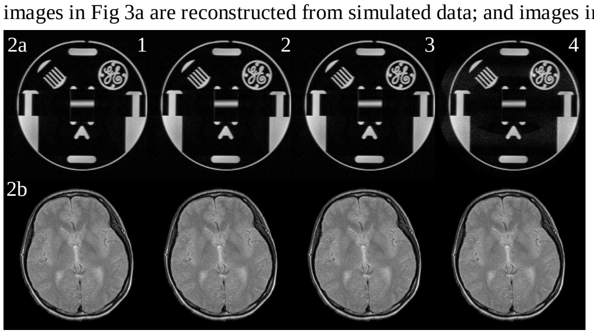
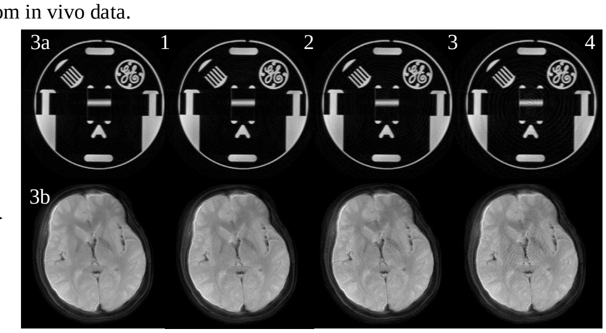


Fig 2 – Images reconstructed using transfer function SENSE with Cartesian trajectories: (a) simulated data; (b) in vivo data.

Fig 3 – Images reconstructed using transfer function SENSE with VD spirals: (a) simulated data; (b) in vivo data.



DISCUSSION: We have shown that fast SENSE reconstruction can be achieved by utilizing the linearity property of the system. In this algorithm, the transfer function is equivalent to a k-space resampling function expressed in a Cartesian coordinate. Comparing to the traditional SENSE reconstruction, the transfer-function based algorithm gains a speed-up factor around $8w^2$ for non-Cartesian trajectories. For example, for the VD spiral trajectory with a reduction factor of two, the transfer-function algorithm takes 3.8s to complete 3 iterations; whereas the gridding based algorithm needs 308.4 s. This computation time is measured based on a Matlab program running on a PC with a 3.06GHz Pentium CPU. The transfer-function algorithm is related to the convolution approach (4) in the sense that both utilize the linearity property. Here, this formulation is treated more generally and the proposed algorithm does not require twice data support.

Although Eq.[1] describes an LSI, the original SENSE implementation with backward and forward gridding results in a shift variant system. The corresponding transfer function is thus location dependent and can be written as $\mathbf{H}(\mathbf{k}, \mathbf{k}')$. In other words, the transfer-function algorithm is not completely equivalent to the gridding approach. Using the transfer-function algorithm, successful image reconstruction has been demonstrated on both simulated and in vivo data. Overall image quality is good, though small residual artifacts remain on in vivo images with high reduction factors.

ACKNOWLEDGMENTS: NIH-1R01NS35959, NIH-1R01EB002771, Lucas Foundation, Center of Advanced MR Technology of Stanford (NCRR P41 RR 09784)

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