Orthogonal tensor decomposition for analysis of DTMRI anisotropy

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Introduction: Previous work has distinguished isotropic and anisotropic tissue structure through the use of invariant tensor measures such as fractional anisotropy (FA) and relative anisotropy (RA)[1]. While these metrics quantify one measure of anisotropy they do not specifically characterize the *mode of anisotropy*[2]. In general, anisotropy is an umbrella term for tensors whose *mode* can range from linearly anisotropic (one large eigenvalue, two small) to orthotropic (three distinct eigenvalues) to planar anisotropic (two large eigenvalues, one small), each of which potentially identifies unique neuroanatomy. The *magnitude of anisotropy* characterizes the degree to which a tensor exhibits a particular *mode of anisotropy*. Both the *magnitude* and the *mode of anisotropy* can be used to generate scalar colormaps of DTMRI data to provide insight to the underlying tissue structure.

Theory: Diffusion tensors are 3x3 real-valued symmetric positive definite tensors and can be decomposed into three eigenvalues and three eigenvectors. The proposed metrics are formulated in terms of standard tensors operators, but for completeness the metrics are also cast in terms of the eigenvalues. The first three central moments of the eigenvalues are μ_1 (bulk mean diffusivity), μ_2 (related to the magnitude of the anisotropy); and μ_3 (determines the *mode of anisotropy*).

$$\mu_1 = \frac{1}{3}\sum_{i=1}^{3}\lambda_i , \qquad \mu_2 = \frac{1}{3}\sum_{i=1}^{3}(\lambda_i - \mu_1)^2 , \qquad \mu_3 = \frac{1}{3}\sum_{i=1}^{3}(\lambda_i - \mu_1)^3$$

Any tensor D can be decomposed into a sum of isotropic ($\bar{D}$) and anisotropic ($\tilde{D}$) components, $D = \tilde{D} + \bar{D}$. $\tilde{D}$, also termed the deviatoric component, is defined as $\tilde{D} = D - \frac{1}{3}tr(D)I$, and the isotropic component $\bar{D} = \frac{1}{3}tr(D)I$. The isotropic and anisotropic tensors can be further decomposed into a magnitude and mode. The magnitude of any tensor is defined as $mag(A) = \sqrt{A:A} = \sqrt{tr(AA^T)}$. It can be shown that $mag(\tilde{D}) = \sqrt{3\mu_2}$, $mag(\bar{D}) = \sqrt{3}\mu_1$, and $mag(D) = \sqrt{3\mu_1^2 + 3\mu_2}$. Thus, each of the tensor magnitudes is related to the first and second central moments of the eigenvalues and is defined over the interval $[0,+\infty)$.

The mode of a tensor is defined as $mode(A) = \det(A / mag(A))$. The mode of the isotropic component of the tensor is constant; there is only one mode of dilatation: $mode(\bar{D}) = \det(\bar{D}/mag(\bar{D})) = \det((1/\sqrt{3})I) = 3^{-3/2}$. The *mode of anisotropy* is found from $mode(\tilde{D}) = \det(\tilde{D}/mag(\tilde{D}))$ and by definition it can be shown that $mode(\tilde{D}) \in [-1/3\sqrt{6}, 1/3\sqrt{6}]$. Furthermore $mode(\tilde{D})$ is related to the third central moment of the eigenvalues by $mode(\tilde{D}) = \mu_3/(3\mu_2)^{3/2}$. When $mode(\tilde{D}) = -1/3\sqrt{6}$ the tensor is linearly anisotropic; when $mode(\tilde{D}) = 0$ the tensor is orthotropic; when $mode(\tilde{D}) = 1/3\sqrt{6}$ the tensor is planar anisotropic. $mode(\tilde{D})$ is a distinct measure of diffusion anisotropy suitable for the assessment of neuroanatomy. It can be shown that $mag(\bar{D})$, $mag(\tilde{D})$, $mode(\tilde{D})$ are mutually orthogonal and are consistent with previously described orthogonal metrics [3].

Basser [1] defined two common anisotropy metrics. RA is defined as the ratio of the magnitude of the anisotropic part of D and the magnitude of the isotropic part of D and FA is defined as proportional to the ratio of the magnitude of the anisotropic part of D to the magnitude of D. Both are related to the central moments.

$$RA = mag(\tilde{D})/mag(\bar{D}) = \sqrt{\mu_2}/\mu_1 \qquad\qquad FA = \sqrt{3/2}\,mag(\tilde{D})/mag(D) = \sqrt{3\mu_2/2(\mu_2 + \mu_1^2)}$$

The explicit dependence of RA and FA upon μ_1 and μ_2 demonstrates that these measures are related to the mode of isotropy and the magnitude of the anisotropy, but not the mode of anisotropy. As noted by Bahn [3], no single scalar measure can fully characterize tensor anisotropy. This is due to its fundamental constituency of both a magnitude and a mode. Thus, $mode(\tilde{D})$ usefully complements the otherwise incomplete description of anisotropy given by FA or RA alone.

Results: To provide a qualitative assessment of the $mag(\bar{D})$, $mag(\tilde{D})$, and $mode(\tilde{D})$ relative to RA and FA colormapped images of a slice of human DTMRI are shown in Figure 1. Informative scalar colormaps are made from the individual metrics and from combinations of the same (Figure 1). For example, $mode(\tilde{D})\langle FA\rangle$ produces an scalar map with the intensity of FA but with color coding that reveals the underlying *mode of anisotropy*. In Figure 1H $mode(\tilde{D})$ in the corpus callosum indicates linear anisotropy. In contrast, a region of planar anisotropy delineates the boundary between the corpus callosum and the cingulum bundle (see arrow).

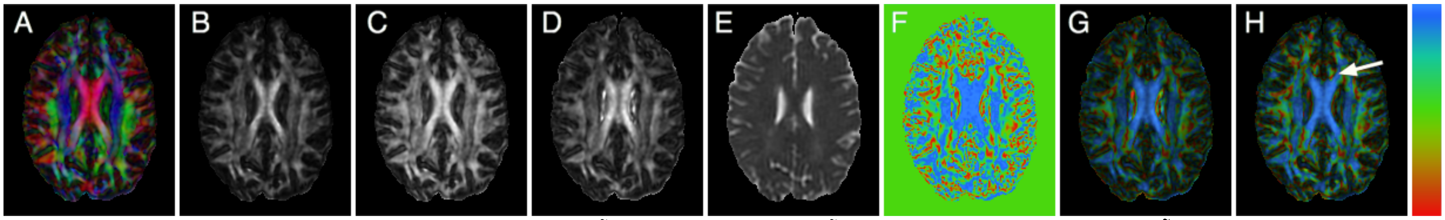

Figure 1. A: RGB of principle eigenvector, **B:** RA, **C:** FA, **D:** $mag(\tilde{D})$ **E:** $mag(\bar{D})$ **F:** $mode(\tilde{D})$, **G:** $mode(\tilde{D})\langle mag(\tilde{D})\rangle$ **H:** $mode(\tilde{D})\langle FA\rangle$. The color indicates $mode(\tilde{D})$ Blue– linearly anisotropic, Red–orthotropic, Green–planar anisotropic and intensity modulation is produced through the function within the $\langle\,\rangle$.

Discussion: The fact that RA and FA do not depend upon the third central moment underlies their lower noise sensitivity and is perhaps the key to their popular and robust use. The use of the third central moment to define a metric of anisotropy is likely to be more sensitive to noise as the metric depends upon the cube of eigenvalue differences. Future work needs to characterize the noise sensitivity of the *mode of anisotropy*. The new tensor decomposition offers a way to assess the anisotropy of a tensor. Although another orthogonal decomposition has been proposed [3] the advantage of the basis contained herein is the use of conventional tensor operators and the avoidance of the need to explicitly calculate the eigenvalues. Furthermore, it has been observed that $mag(\bar{D}) = \sqrt{3}\mu_1$ is relatively constant in healthy parenchyma, hence its use within the orthogonal decomposition is potentially advantageous as anatomical variation is characterized by the two remaining orthogonal components $mag(\tilde{D})$ and $mode(\tilde{D})$. Westin [4] proposed a tensor decomposition that provided measures of linear and planar anisotropy, but without a convenient measure of orthotropy, nor were the metrics mutually orthogonal.

Conclusions: $mag(\bar{D})$, $mag(\tilde{D})$ and $mode(\tilde{D})$ form an orthogonal basis for tensor decomposition and provide a basis for unique measures of anisotropy. This description of anisotropy may provide improved insight for DTMRI analysis. These metrics do not exclude FA and RA in a complete analysis of DTMRI data. As shown in Figure 1H, for example, the combination of anisotropy metrics allows the simultaneous display of the $mode(\tilde{D})$ and the FA.

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