The Lagrangian Multiplier Method as a Basis for Elastic Image Registration

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Synopsis

Non-rigid body registration is important for objects that undergo configuration change during observation. Previous non-rigid registration approaches require an arbitrary free parameter. We report on an approach in which the registered state is considered an equilibrium configuration, for all forces acting on the system, and the single parameter is equated with the value of a Lagrangian multiplier (L), that guarantees stability of the system. The boundary constraint energy tends to zero for a unique value of L. The functional dependence of the Lagrange multiplier is shown to be consistent with established optimality criteria.

Introduction

Registration of time series data is important for reliable estimates of physiological parameters in organs, such as the heart, where observed configuration changes can be comparable to rigid body motion. Davatzikos et al. reported a non-rigid registration method based on a material (elastic) model (1), however their approach involved a free parameter that had to be chosen. In the present method, we investigated an elastic registration approach based on a variational mechanics principle, the Lagrangian multiplier method (2), in which the single control parameter and the cost function, arise naturally from solution of an equilibrium problem, with an explicit auxiliary, boundary constraint.

Method

Theory: Consider intrinsic energy of the body, related to internal forces, to be the strain energy. For an isotropic, homogeneous, incompressible elastic thin plate, this strain energy, U, is given by; $U = G\oiint_A [(\varepsilon_x^2 + \varepsilon_y^2) + \frac{1}{2}\gamma_{xy}^2]dA$, where, ε_p is the normal-strain in the p direction, γ_{pq} is the shear-strain in pq plane, and G is the shear modulus.

We consider the problem of registration to involve a total "system", consisting of internal energy, plus auxiliary conditions (in this case, constraining energy provided by the bounding contours). In the equilibrium state the bounding contour gives rise to a constraint (shown as "C", in the cartoon in **Figure 1**), which, essentially, maps correspondence points on the source boundary to the target boundary. This can be written in discrete form as an energy functional, U_f ; $U_f = G\sum_k\sum_l[(X(k,l) - f^X(k,l))^2 + (Y(k,l) - f^Y(k,l))^2] = 0, \quad (k,l) \in B$. Here, (k,l) is the pixel set belonging to the contour, B, and $(f^X(k,l), f^Y(k,l))$ are the pixel positions of the contour of the target image, obtained from a mapping, f, of the pixel (X(k,l), Y(k,l)) of the contour B' of the source image.

The Langrangian multiplier method takes advantage of the freedom allowed by the boundary constraint to introduce a control parameter, L, such that we have a new "system" potential energy function; $\overline{U} = U + L \quad U_f$.

In the registered state the total system (under internal forces plus boundary forces); is considered to be in equilibrium, i.e. an energy minimum, or; $\delta\overline{U} = 0$.

We can gain an understanding of the general properties of the system, in terms of values assumed by the Lagrangian multiplier, by an iterative solution, in discrete form. By defining strains and shear in discrete form, and differentiating the resultant expression for $\overline{U}$, we get an expression for the material coordinates in terms of the Lagrangian multiplier;

$$X(i,j) = \frac{X(i+1,j) + X(i-1,j) + X(i,j+1) + X(i,j-1) + L\cdot I(i,j)f^X(i,j) + Y(i+1,j) - Y(i,j) - Y(i+1,j-1) + Y(i,j-1)}{4 + L\cdot I(i,j)}$$

$$Y(i,j) = \frac{Y(i+1,j) + Y(i-1,j) + Y(i,j+1) + Y(i,j-1) + L\cdot I(i,j)f^Y(i,j) + X(i,j+1) - X(i,j) - X(i-1,j+1) + X(i-1,j)}{4 + L\cdot I(i,j)}$$

where (X(i,j),Y(i,j)) and (x(i,j),y(i,j)) are the pixel in the target and reference images, respectively. I(i,j) is unity if (i,j) belongs to the set B, and 0 elsewhere.

This expression then permits us to graph physically intuitive quantities, such as the total deformation (average motion of each pixel between the target and source images; $D = \frac{1}{N}\sum_i\sum_j\sqrt{(\mathbf{X}(i,j) - \mathbf{x}(i,j))^2 + (\mathbf{Y}(i,j) - \mathbf{y}(i,j))^2}$), as a function of values assumed for L, and the relation of this functional, to the well established cost structure, normalized cross-correlation (NCC).

Results

Figure 2 illustrates the variation of Deformation, with respect to values assumed by the Lagrangian multiplier. We also plot a cost function that has the convenient property of being unity, when optimized, **Figure 3**. Note that as we pick a larger and larger number of boundary correspondence points, for both these functionals, there is an apparent convergence to a unique value of the Lagrangian multiplier that optimizes the functional.

Discussion and Conclusion

In the present approach [which we have termed Constrained Energy Minimization Theory (CEMT)], we replace the requirement of assuming values for material constants of the body to be registered, with one in which information that is valid for the bounding contour, at equilibrium, naturally gives rise to the single control parameter of the theory. The optimality criterion also arises as a natural consequence of the equilibrium equation.

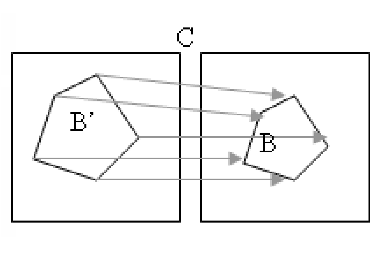

Figure 1: Mapping of boundary correspondence points

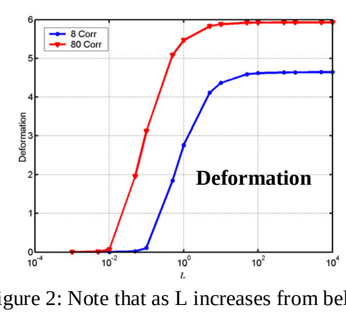

Figure 2: Note that as L increases from below, there is an abrupt maximization of deformation, for $L \sim 10^{-2}$

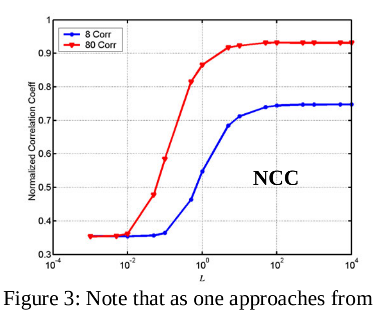

Figure 3: Note that as one approaches from below, normalized cross-correlation (NCC) is maximized for $L \sim 10^{-2}$

References:

1. Davatzikos C, et al. IEEE Transactions Med Imaging; 15:112-115, 1996.
2. Lanczos C, The Variational Principles of Mechanics, 4th Ed. Dover Publications, Inc, NY; 1986.