Cramér-Rao Bounds for 3-Point Dixon Imaging

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Introduction

The noise of the estimates for fat and water in 3-point Dixon imaging [1] depends on many factors including the echo times, the field inhomogeneity, the ratio of fat to water [2] and the reconstruction algorithm [3]. The Cramér-Rao bound (CRB) is the lower bound on the variance of any unbiased estimate of the fat and water. It provides a measure of the minimum uncertainty of the desired quantities for a given data acquisition, independent of the reconstruction algorithm. The expression of the bound itself provides insight into how the different components of the imaging chain affect the noise performance. The CRB can be used to choose echo times and reconstruction algorithms. It arises naturally when considering MRI as a statistical estimation problem.

Background

The CRB is an inequality of the Fisher Information Matrix (FIM) which sets a lower bound on the covariance of any unbiased estimator. When comparing matrices "≥" means that the difference of the two matrices is positive semi-definite. In particular it implies that the variance of the estimates (diagonal terms of the covariance matrix) cannot be smaller than that of the inverse of the FIM. To present the results in terms of the Number of Signal Averages (NSA), we normalize the variance of the estimate by the variance of the original measurements. To compute the CRB all we need is the probability model for generating the measurements (independent Gaussian). There is no specification of the estimation algorithm. One can view the FIM, and equivalently the CRB, as a sensitivity of the data to the parameters being estimated.

Fisher Information Matrix (FIM)

$$F_{ij} = -\left\langle \frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j} ln Pr(s|p) \right\rangle$$

s : vector containing the data

p : vector containing the parameters of the model

$Pr(s|p)$: probability of observing s given p

Cramér-Rao Bound (CRB)

$$C_{\hat{p}} \geq F^{-1}$$

$\hat{p}$: Any unbiased estimator of the parameters

Number of Signal Averages (NSA)

$$NSA_i = \frac{\sigma^2(s)}{\sigma^2(\hat{p}_i)}$$

Maximum NSA for estimating p_i

$$NSA_i \leq \frac{\sigma^2(s)}{[F^{-1}]_{ii}}$$

Model and Expressions for CRB

We assume the signal can be reconstructed on a pixel-by-pixel basis. If we know the field map, the estimation of fat and water is a linear problem. In that case, the CRB of all 4 unknowns (I and Q values for fat and water) is the same and has a closed form solution [3]. For an unknown field map, the estimation is nonlinear. The variance of the fat and water depend on the fat and water densities themselves [2] and they become coupled to the estimate of the field map.

Signal Equation in Complex Notation

$$s_m = (\rho_w + \rho_f\, e^{iTE_m})\, e^{i\,\Psi\, TE_m} + n_m$$

s_m : complex measurement at mth echo

ρ_w : water signal

ρ_f : fat signal

TE_m : mth echo, normalized by chemical shift

Ψ : field inhomogeneity

n_m : noise of mth measurement

Signal Equation in Real Matrix Notation

$$s = A(\Psi)\,\rho + n$$

s : vector of 6 measurements (real and imaginary)

$A(\Psi)$: 6x4 matrix

ρ : vector of 4 signals (real and imaginary)

n : vector of 6 noise signals with variance $\sigma^2(s)$

FIM (4x4) for Field Map Known

$$F = \frac{1}{\sigma^2(s)} A^{\dagger} A$$

FIM (5x5) for Field Map Unknown

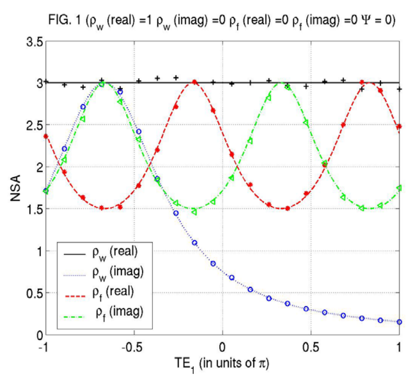
$$F_{1..4,1..4} = \frac{1}{\sigma^2(s)} A^{\dagger} A$$

$$F_{i=1..4,5} = \frac{1}{\sigma^2(s)} \left[A^{\dagger}\frac{\partial A}{\partial \Psi}\rho\right]_i$$

$$F_{5,5} = \frac{1}{\sigma^2(s)} \rho^{\dagger}\frac{\partial A^{\dagger}}{\partial \Psi}\frac{\partial A}{\partial \Psi}\rho$$

Results and Discussion

Given our model, the Maximum Likelihood Estimator (MLE) of the fat, water and field map is Nonlinear Least Squares (NLS). We verify the expression for the CRB with Monte Carlo studies and experimental measurements. Fig. 1 shows the variance of all the unknowns as a function of the first echo time (TE1) for an acquisition of the following form: TE2= TE1+ 2π/3 and TE3 = TE2 + 2π/3 (where TEi is the echo time in units of phase rotation between fat and water) and for the case where only water is present. If the field map was known, the NSA would equal 3 for all unknowns and be independent of TE1. The deviation from this behavior comes from the nonlinearity of the estimation problem. The optimal TE1 results in a symmetric acquisition with TE2=0. Note that the when the estimation is not linear, the variance of the real and imaginary components of the signal are not equal. In Fig. 2, we verify that the poor behavior seen in symmetric acquisitions when the fat and water signals are equal [2] is intrinsic in the measurements, not a consequence of the reconstruction algorithm.



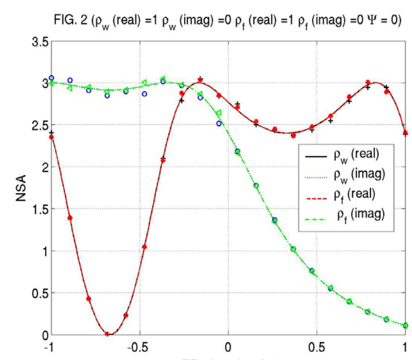
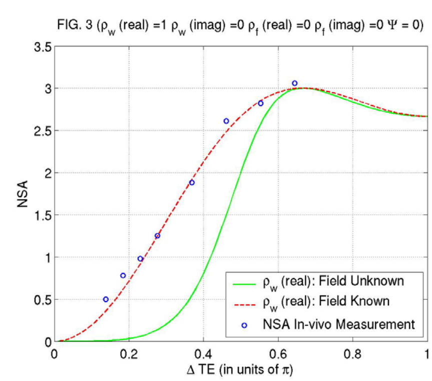


Figure 1,2: Solid lines (CRB), Symbols (Monte Carlo); Figure 3: Symbols (Experimental)

In Fig 3 we present the results from a symmetric acquisition (where the second echo is on resonance): TE1 = -ΔTE, TE2 = 0, TE3 = ΔTE for a situation where there is only water in the pixel. The difference between the two solid lines quantifies the information loss from not knowing the field map. The NSA of the field map (not shown) increases with ΔTE which explains the decreased difference between the two curves decreases as ΔTE increases. This plot can also be used to make trade-offs between artifacts and noise for sequences like SSFP that are susceptible to artifacts for large echo times. The In-vivo measurements were obtained from a series of cardiac images of a normal volunteer by using a ROI in the cardiac wall in the reconstructed water image and using an ROI outside of the object for the source variance. The curves in Fig. 3 show the CRB for cases in which the field is known and unknown. The experimental values are for the algorithm presented in [3] with the field map unknown. The results are at first surprising in that the experimental values are better than the CRB for the case where the field map is known. However, the algorithm used for Fig 3 incorporated field map smoothing in the iterations of the NLS algorithm [3]. For areas where the field is slowly varying, field map smoothing reduces the uncertainty introduced by needing to estimate the field map to the point where it matches the case where the field map is known. For this model, in terms of NSA, there may be other algorithms that match the performance of the algorithm in [3] but none can do better.



References 1. Glover et al, JMRI; 1991; 1:521-530. 2. Wen et al, ISMRM, 2003, pg 483, 3. Reeder et al, MRM, *in press*.

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