A Rigorous Method for Calculus on Diffusion Tensors: Taking into Account Positivity of Eigenvalues

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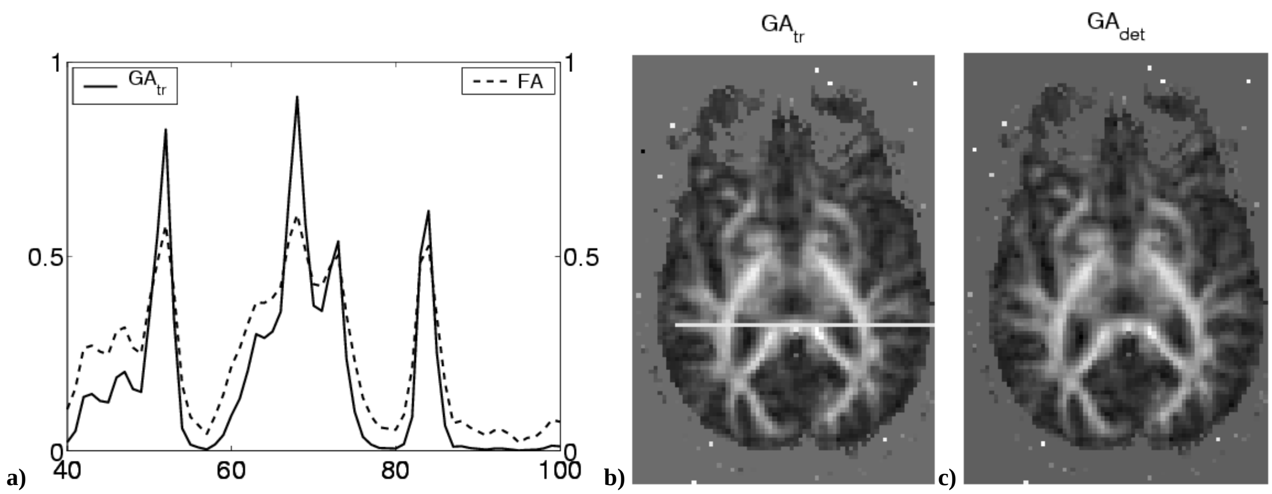
Introduction. Diffusion tensors from Diffusion Tensor MRI (DT-MRI) are represented as matrices D with *positive eigenvalues* in an orthogonal coordinate system. It enforces the *physical constraint* that there are no directions with negative diffusivity. Common diffusion measurements (Fractional Anisotropy FA, means, etc.) do not explicitly consider the requirement for positive eigenvalues, which we argue can lead to flawed or counter-intuitive interpretations. We propose here to use a definition of distance on the space of all diffusion tensors that explicitly takes into account this positivity. We derive from that as an example a new, formally justified, anisotropy measure. Furthermore, we describe how this space of diffusion tensors allows a formal means of computing the mean of a set of diffusion.

Theory. From differential geometry, we get a natural geometry on the space of symmetric real matrices with positive eigenvalues λ_i [TerrasII].

Distance. Explicitly, the infinitesimal change of length for a change of tensor *dD* is given by $ds^2 = \mathrm{trace}(D^{-1}dD)$. The induced distance from *D* to the identity tensor *I* is dist (*I*,*D*)= $(\sum \log \lambda_i^2)^{1/2}$ and in general dist (D_1, D_2) = $\mathrm{dist}(I, D_1^{-1/2} D_2 D_1^{1/2})$.

Anisotropy. We can give a principled definition of anisotropy, as the geodesic distance of the tensor *D* to an isotropic tensor with diagonal values given by either $\langle D \rangle = \mathrm{trace}(D)/3$ or $|D| = \det^{1/3}(D)$. We label the latter geodesic anisotropy as $GA_{\det}$ and the former as GA_{tr}, where $GA_{tr} = (\log^2(\lambda_1/\langle D \rangle) + \log^2(\lambda_2/\langle D \rangle) + \log^2(\lambda_3/\langle D \rangle))^{1/2}$. $GA_{\det}$ is arguably a slightly better choice mathematically, although the trace is more familiar to the DT-MRI community. Many measures of anisotropy are already available, the *FA* being the most common [basser98]. The *FA*, given by $\|D - \langle D \rangle I\| / \|D\|$ has a noticeable analogy with the geodesic anisotropy in that it represents the relative Euclidean distance to its isotropic part. But *GA* tends to infinity when an eigenvalue approaches zero, as desired, where the *FA* of the (totally unrealistic) tensor diag ([1,0,0]) is 0.82, i.e. not very different to that measured in the Corpus Callosum. A further important concept that can now be defined rigorously in this space is the mean of diffusion tensors. The mean $\overline{D}$ of a set of tensors with positive eigenvalues is defined by $\sum_k \log(D_k^{-1}\overline{D}) = 0$, from which follows for example that the mean of D_1 and D_2 is $\overline{D} = D_1(D_1^{-1}D_2)^{1/2}$.

Experimental Results. The figure shows a comparison of *FA* with both *GA* measures (Siemens 1.5T, 20 directions b = 1000 $\mathrm{s/mm}^2$) on a typical DT-MRI dataset. a) shows an intensity profile from the position indicated by the white line in b). Note the increased contrast in *GA*. c) shows a $GA_{\det}$ map corresponding to the GA_{tr} map shown in Fig b).



Discussion-Conclusion. The example of diag ([1,0,0]) illustrates a limitation of current measures (such as *FA*) in characterizing anisotropy. This example shows that the contrast in *GA* and *FA* will be different, and *GA* will be more sensitive to highly anisotropic tensors.Taking account of the positivity of eigenvalues, as in the proposed measure of Geodesic Anisotropy, therefore has potentially important implications in quantitative diffusion analysis. In addition, when a combination of tensor data is required (e.g. for SNR improvement), we have shown how a rigorous mean of a set of tensors can be calculated while incorporating the constraint of positive eigenvalues.

References

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