Higher Order Tensor Analysis and Higher Order SVD for Diffusion Tensor Analysis

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Introduction. The standard Singular Value Decomposition (SVD) is one of the main tools to study classical DT-MRI data, when they are represented as symmetric, positive definite tensors of order 2. Much current research in diffusion MR, however, is directed towards Higher Angular Resolution Diffusion Images (HARDI) data, that can be modeled in a number of different ways, e.g. as multi-exponential [1] or a natural generalization as Higher Order Tensor (e.g. [2]) to take account of multiple fibre directions within individual voxels. In this context, a generalization of SVD to such data would appear highly desirable, and such generalizations, called Higher Order SVD (HOSVD) although not commonly used, do exist [3]. Thus, our aim here is to introduce and investigate the usefulness of HOSVDs in HARDI, and describe mathematically the transition from 2^{nd} order tensor to higher data.

Theory-Methods. Multiexponentials: Physically, a voxel can contain multiple contributions to diffusion due to partial volume effect. The most straightforward description of the measured signal for a given diffusion gradient orientation $\boldsymbol{g}$ is to assume that it is a linear combination of the signals from individual Gaussian diffusion [1]. This leads to a multi-exponential description of the signal: $s(\boldsymbol{g})/s_0 = f \exp(-b\boldsymbol{g}^t\boldsymbol{D_1}\boldsymbol{g}) + (1\text{-}f)\exp(-b\boldsymbol{g}^t\boldsymbol{D_2}\boldsymbol{g})$. The factors f and $1\text{-}f$ measure the size of each contribution; b-factors encode the amount of diffusion weighting. As a first theoretical result, this signal is asymptotically weighted average of the tensors (this can be seen from a Taylor expansion in b):

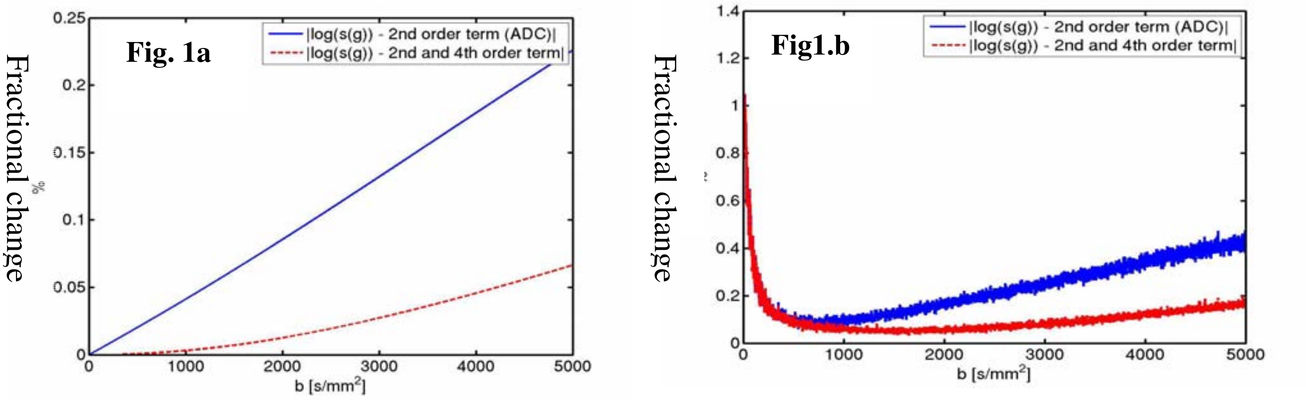
$$\log(s(\boldsymbol{g})/s_0) \approx -b\,\boldsymbol{g}^t\boldsymbol{D}\boldsymbol{g} \approx -b\boldsymbol{g}^t(f\boldsymbol{D_1} + (1\text{-}f)\boldsymbol{D_2})\boldsymbol{g} + \text{(higher order terms)}, \qquad (1)$$

The fibre direction is extracted as the principal eigendirection of the tensor $\boldsymbol{D}$. When b becomes larger, the expansion to second order becomes inaccurate, and higher order (even) terms, of at least 4^{th} order have to be considered (expressions of the type $\boldsymbol{D}_{ijkl}$). This turns out to have a surprisingly simple expression for two compartments: the coefficient of b^2 in the Taylor expansion, *i.e.*, the

$$4^{th}\text{ order tensorial term in (1)} \approx f(1\text{-}f)(\boldsymbol{g}^t(\boldsymbol{D_1} - \boldsymbol{D_2})\boldsymbol{g})^2 \,. \qquad (2)$$

4^{th} Order Stejskal-Tanner: An alternative view is the one proposed in [2], where instead of treating $s(\boldsymbol{g})$ as a multi-exponential, it is viewed directly as an exponential of higher order tensors, $\log(s(\boldsymbol{g})/s_0) \sim -b\,\boldsymbol{D}_{ijkl}\,\boldsymbol{g}_i\,\boldsymbol{g}_j\,\boldsymbol{g}_k\,\boldsymbol{g}_l$ (summation over repeated coordinate indices, this is eq. (11) in [2]). Thus, in both cases tensors of at least 4^{th} order are needed.

HOSVD: Standard SVDs can be described as a decomposition of the tensor in a weighted sum of rank 1 terms (dyadic products). Two generalizations of this interpretation of the SVD to expressions of the form $\boldsymbol{D}_{ijkl}$ are available: the PARAFAC and Tucker-HOSVD (see [3] for details). As a simulation, we construct a pure 'crossing' situation of two tensors $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$ with orthogonal principal directions forming an angle of 90° (arbitrary non-principal directions, both with FA= 0.86, f=0.75).



Results. Multi-exponential view: Figure 1a shows the relative differences $|\log(s(\boldsymbol{g})/s_0) - (2^{nd}\text{ order term})|/|\log(s(\boldsymbol{g})/s_0)|$, and $|\log(s(\boldsymbol{g})/s_0) - (2^{nd}\text{ and }4^{th}\text{ order terms})|/|\log(s(\boldsymbol{g}/s_0))|$, as a function of b, where $\boldsymbol{g}$ is taken among 30 icosahedral directions . **Fig.1a**, without added noise, **Fig.1b**, with a b-dependent SNR corresponding to fixed std of the noise (fixed SNR from 100 at b=0 to 45 at b=5000). The unrealistically high SNR is chosen to ease visualisation in the graphs; the effect is similar but more marked at lower SNRs.

4^{th} order Stejskal-Tanner: We construct a 4^{th} order tensor as the *tensor product* (Kronecker, dyadic) $\boldsymbol{D_1} \times \boldsymbol{D_2}$ to which we add increasing amounts of random error up to 10% of the size of the tensor (horizontal axis in **Fig.2**). We then compute the error of the directions found by the HOSVD with the original directions, (**Fig. 2**, vertical axis) as the sum of the norms of the cross-product between directions (unit vectors, thus the norm is the sine of the angle, approx. the angle in radians for small values) found by HOSVD and the gold-standard. This quantity is zero for perfect alignment. The two HOSVDs give identical results for this case.

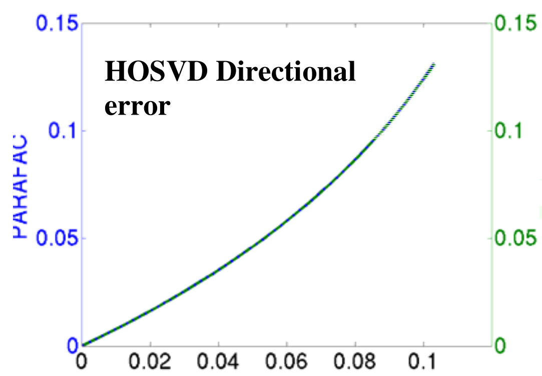


Fig. 2: Error ratio

Discussion: At 'low' b-values, the 2^{nd} order tensor corresponding to a mixture of signals is asymptotically the averaged tensor, with good approximation up to b~1000 without noise, but the noise dominates at very low b-values (cf. **Fig 1b**). Note that from the point of view of this approximation, values around b=1000 are nearly optimal (which agrees with [4]); any method which attempts to extract multiple directions for low b-values does it from the discrepancy between fitted and predicted in **Fig1a-b**. This is considerably improved by the correct 4^{th} order fit for a multi-exponential. For higher b-values, the HOSVDs could provide a simple and efficient way to extract the tensor direction from a tensor, provided the tensor expresses multiple directions in the way described by the 4^{th} order Stejskal-Tanner equation, *i.e.*, the approach in [2] rather than multi-exponentials. Note that to apply the HOSVD to a multiexponential directly, we would need to extract the crossing term $\boldsymbol{D}_1\boldsymbol{D}_2$ from the (symmetric) tensor product of $\boldsymbol{D}_1\text{-}\boldsymbol{D}_2$ with itself, this is $\boldsymbol{D}_1^2 + \boldsymbol{D}_2^2 - 2\boldsymbol{D}_1\boldsymbol{D}_2$ thus is a mixture.

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[4] Jones et al, *MRM*, **42**, 515-525.