Accelerated Exponential Fitting for Rapid Relaxation Time Mapping

H. Eggers[1], P. Boesiger[2]

[1]Sector Technical Systems, Philips Research, Hamburg, Germany, [2]Institute for Biomedical Engineering, University and ETH Zurich, Zurich, Switzerland

Introduction

Mapping of relaxation times such as T_1 and T_2 generally involves the fitting of an exponential function to the evolution of the signal intensity in each pixel of a time series of images. The prevalent method employed for this purpose, the Levenberg-Marquardt method, pursues a non-linear least squares approach [1]. For applications that require a rapid mapping, its high computational complexity is a limiting factor, however. Other, faster methods have been devised, but they prove less accurate. For instance, a linear regression leads to an unstable estimation in the presence of noise, and a numerical integration to a systematic overestimation [2]. To decrease computational complexity while conserving accuracy, the present work suggests to solve the non-linear regression problem by searching for a real root of a polynomial in a small interval, and it demonstrates this approach on T_2^* mapping.

Methods

Let s_k denote the samples of signal intensity taken at $k\Delta t$, where $k = 0 \ldots N-1$. The best fit of a monoexponential decay to this time series in a least squares sense is given by the minimum of the error function

$$\varepsilon(c,q) = \sum_{k=0}^{N-1} (s_k - c\,q^k)^2 \,, \qquad \text{with} \quad q = e^{-\lambda \Delta t} \,.$$

For finite positive relaxation times, q is confined to the interval (0,1). Setting the partial derivatives of ε with respect to c and q to zero and eliminating c yields

$$p(q) = p_1(q)\,g_1(q) - p_2(q)\,g_2(q) \,,$$

where

$$p_1(q) = \sum_{k=1}^{N-1} s_k\,k\,q^k \,, \qquad p_2(q) = \sum_{k=0}^{N-1} s_k\,q^k \,,$$

$$g_1(q) = \frac{1-q^{2N}}{1-q^2} \,, \qquad g_2(q) = \frac{1}{1-q^2}\left(\frac{q^2-q^{2N}}{1-q^2} - (N-1)\,q^{2N}\right).$$

Finding the best fit is thus reduced to searching for a real root of the polynomial $p(q)$ in the interval (0,1). Among the various methods available for such a search [3], the Newton-Raphson method is one of the most efficient. It requires the evaluation of both p and its first derivate p' for arbitrary q. The latter may be expressed as

$$p'(q) = p_3(q)\,g_1(q) + 2\,p_1(q)\,g_2(q)/q - p_1(q)\,g_2(q)/q - p_2(q)\,g_3(q) \,,$$

where

$$p_3(q) = \sum_{k=0}^{N-2} s_{k+1}(k+1)^2\,q^k \,, \quad g_3(q) = \frac{2}{q(1-q^2)}\left(2g_2(q) - g_1(q) - (N-1)^2 q^{2N} + 1\right).$$

In this way, the computation of p and p' for one q essentially involves the calculation of p_1, p_2, and p_3 only, which amounts to about $9N$ floating point calculations per iteration. An initial guess for q is, for instance, obtained from any two samples of the time series.

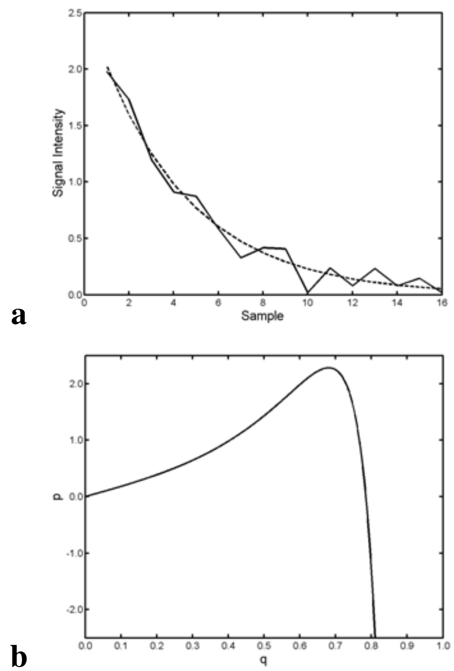

*Fig. 1. **a:** Simulated input data (solid) and fitted exponential decay (dashed). **b:** Corresponding polynomial p(q) with one real root in the interval (0,1).*

Results

Fig. 1 illustrates this approach for an ideal monoexponential decay with added noise. The corresponding $p(q)$ shows exactly one real root in the interval (0,1). Starting with the inverse ratio of the first two samples, the deviation from the result of the Levenberg-Marquardt method was less than 0.2% after only two iterations.

The application of this approach to a series of 30 brain images, which were acquired with a multi-gradient echo sequence (ΔTE = 1.9 ms), to map T_2^* confirmed that a small number of iterations is sufficient to attain a high accuracy, even in the presence of significant noise. Fig. 2 shows the decrease in error with increasing number of iterations in this case, using again the results of the Levenberg-Marquardt method as reference. By comparison, the numerical integration approach yielded an error of about 10^{-1}.

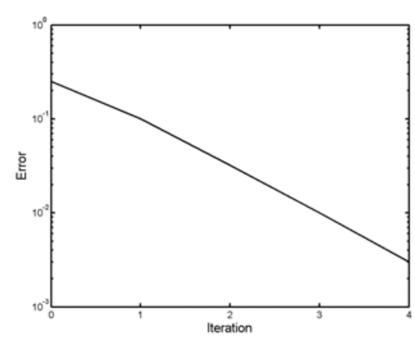

Fig. 2. Progression of the average relative error in the T_2^* estimation as a function of the number of iterations.

Discussion

The shape of $p(q)$ in Fig. 1b was found to be typical. It suggests to preferably start with an overestimation of q to exploit the more rapid variation of p for higher q and to prevent a convergence towards $q = 0$. Such an overestimation is, among others, provided by the numerical integration approach.

Conclusions

The described root finding approach is applicable to the fitting of a monoexponential function to an equidistantly sampled time series of signal intensities. It achieves essentially the same accuracy as the Levenberg-Marquardt method, but requires only a fraction of the calculations. Hence, it appears particularly suited for the real-time quantification of relaxation time changes. Whether this approach can be adapted to more complex models of relaxation remains to be investigated.

References

1. Marquardt DW. SIAM J Appl Math 1963; 11:431-441.
2. Hagberg GE, et al. Magn Reson Med 2002; 48:877-882.
3. Press WH, et al. Numerical recipes in C. Cambridge: Cambridge University Press; 1999. 994 p.