The Theoretical Basis of Rectangular Magnet and Coil Design

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Introduction

A standard mathematical approach to the design of magnets, gradient coils, shim coils etc. is first to expand the field from an elementary current in a basis set. That set, comprising solutions of Laplace's equation, is dependent on the coordinate system chosen and for most applications, cylindrical polar or spherical polar coordinates are appropriate. Once the elementary field expansion is known, one may then, with a variety of strategies, extend the analysis to distributions of current and solve the inverse problem of finding the current distribution that generates a so-called "target field".

Occasionally, use of a Cartesian coordinate system would be helpful. For example, a "magnetic wall" that creates a homogeneous field over a cuboidal volume (width 50 cm, height, depth 20 cm) in front of it would be advantageous for "walk-up" mammography. However, to quote Morse and Feshbach (1), the Greens' function – the basis of the current determination – is in "a particularly obdurate form" in Cartesian coordinates and the desired expansion is not forthcoming. Nor do standard texts on electrodynamics, e.g. "Jackson", give solutions for this problem and the author has found no solution in the literature. Thus a solution is given here.

The Field Due to a Point Current Source

Let there be a current at point Q(x_0, y_0, z_0) given by $\mathbf{J}dV$ where $\mathbf{J}$ is vector current density and $dV = dx_0\, dy_0\, dz_0$ is an elementary volume. Then at some point of interest P(x, y, z), the magnetic field d**B** is given by the Biot-Savart law (2) as

$$d\mathbf{B} = \frac{\mu_0}{4\pi r^3}(\mathbf{J}\times\mathbf{r})\,dV; \quad r^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \qquad [1]$$

where **r** is the vector joining Q and P. As is usual for magnetic resonance applications, we consider only the z component of **B**. Then

$$\frac{dB_z}{dz_0} = \left(\frac{\mu_0}{4\pi\, r^3}\right)(J_x\eta - J_y\chi)\,d\chi\, d\eta; \quad \chi = x - x_0, \quad \eta = y - y_0, \quad \varsigma = z - z_0 \qquad [2]$$

Let the field derivative be a 2D forward Fourier transform in χ and η. Then, if k_x and k_y are the Fourier-conjugate xy variables, after some algebra, the transform F is:

$$F = \left\{\left(\frac{\mu_0}{4\pi}\right)e^{-ik_x x_0}e^{-ik_y y_0}\right\}\left\{J_x I_1^{\eta} - J_y I_1^{\chi}\right\} \quad I_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{bmatrix}\eta\\ \chi\end{bmatrix}\frac{e^{-ik_x\chi}e^{-ik_y\eta}}{\left(\chi^2+\eta^2+\varsigma^2\right)}\,d\chi\, d\eta \qquad [3]$$

Thus far is obvious; the challenge is the formidable integral. Surprisingly, with algebraic drudgery and two published integrals (2), we may obtain for $z > z_0$:

$$F = \left\{\left(\frac{-i\mu_0}{2k_z}\right)e^{-ik_x x_0}e^{-ik_y y_0}e^{-k_z(z-z_0)}\right\}\left\{k_y J_x - k_x J_y\right\}; \quad k_z^2 = k_x^2 + k_y^2 \qquad [4]$$

where k_z is in accord with Laplace's equation. Importantly, there is a two-dimensional Fourier relationship between F, the transform of the field, and the current density.

We may now follow a well-worn path. Let current flow in a plane of thickness Δz_0 at $z = z_0$ and let a target field be defined in the xy plane. That field is homogeneous over a desired rectangle and eventually goes smoothly to zero outside. As in the current plane div $\mathbf{J} = 0$, it may be shown after further algebra that if H_x, H_y are the inverse 2D Fourier transforms of currents J_x and J_y, and G is the inverse 2D Fourier transform of the target field,

$$H_x = \left\{\left(\frac{2i}{\mu_0}\right)\frac{k_y}{k_z}e^{k_z(z-z_0)}\right\}G; \quad H_y = \left\{\left(\frac{-2i}{\mu_0}\right)\frac{k_x}{k_z}e^{k_z(z-z_0)}\right\}G \qquad [5]$$

In summary to obtain the current to produce a desired field, we define a target field in the xy plane, inverse-transform it, multiply that transform by k-space weighting functions to obtain H_x, H_y and then forward transform to obtain current densities J_x and J_y.

Discussion

There are two important points to note when applying Eqs.[5]: First, the positive exponent $k_z(z - z_0)$. Unless G, the transform of the field, diminishes to zero faster than the exponential rises, functions H are not convergent in k-space and no result can be obtained. One cannot expect distant windings to produce a field limited to a small region. Second, homogeneity of field in the xy plane does not guarantee homogeneity in z; rather, by Laplace's equation, it means that $\partial^2 B_z/\partial z^2 = 0$ and so $B_z = B_0 + G_z z$. As the field in the xy plane is always the same target field, regardless of the position of the current plane, this means that variation of planar current amplitude with z_0 cannot affect the field dependence on z. Rather, *it is the form of the roll-off of the target field* that determines the gradient strength G_z. Fortunately, checking for a suitable roll-off is relatively easy. Setting $\partial B_z/\partial z = 0$, as G is the inverse Fourier transform of B_z, we may then shown that along the z axis we must strive to attain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_z G\, dk_x dk_y = 0 \qquad [6]$$

This approach can be extended to higher orders of variation with z. The figures below show one of many scenarios that fulfil this criterion.

References

1. P. M. Morse and H. Feshbach, Methods of Theoretical Physics, McGraw-Hill, New York, 1953, p. 1254.
2. I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, New York, 1980, p. 426 and p. 736.