Major Speedup of Reconstruction for Sensitivity Encoding with Arbitrary Trajectories

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Introduction

Sensitivity encoding (SENSE) with receiver coil arrays was recently proposed for enhancing imaging speed in MRI [1]. While relatively easy with Cartesian sampling, SENSE has been difficult with general k-space trajectories due to enhanced complexity of the reconstruction. For efficient SENSE reconstruction from arbitrary k-space trajectories, several iterative reconstruction algorithms have recently been proposed [2, 3, 4]. As a common feature, in each iteration step these methods require matrix-vector multiplications, which largely determine the reconstruction speed. In this work we propose a novel implementation of the specific matrix-vector multiplications that occur in SENSE reconstruction, enabling substantial further increases in reconstruction efficiency.

Method

SENSE reconstruction from arbitrary k-space trajectories was formally described in [1]. By neglecting noise statistics or decorrelating receiver noise as proposed in [4], the reconstruction problem reduces to

$$\mathrm{E}^H\mathrm{E}\mathbf{I} = \mathrm{E}^H\mathbf{m}, \tag{1}$$

where $E_{(\gamma,\kappa),\rho} = s_\gamma(\mathbf{r}_\rho)e^{i\mathbf{k}_\kappa\cdot\mathbf{r}_\rho}$, with $s_\gamma(\mathbf{r}_\rho)$ the sensitivity of coil γ at position $\mathbf{r}_\rho$. ρ indexes the voxels and κ the k-space sampling positions, respectively. $\mathbf{I}$ is the image to be reconstructed from the measured k-space data $\mathbf{m}$.

In iterative solving of Eq. 1, the key procedures are the calculations of $\mathrm{E}^H\mathbf{m}$, which is constant during iteration, and $\mathrm{E}^H\mathrm{E}\mathbf{I}^{(n)}$, with $\mathbf{I}^{(n)}$ the reconstructed image at iteration n. As pointed out in [2, 4], $\mathrm{E}^H\mathbf{m}$ can be written as

$$[\mathrm{E}^H\mathbf{m}]_\rho = \sum_\gamma s^*_\gamma(\mathbf{r}_\rho)\left(\sum_\kappa e^{-i\mathbf{k}_\kappa\cdot\mathbf{r}_\rho}m_{\gamma,\kappa}\right), \tag{2}$$

where the last sum can be evaluated efficiently using gridding.

We will now turn attention to the evaluation of $\mathrm{E}^H\mathrm{E}\mathbf{I}^{(n)}$. In [4] $\mathrm{E}^H\mathrm{E}\mathbf{I}^{(n)}$ is evaluated in two steps as $\mathrm{E}^H(\mathrm{E}\mathbf{I}^{(n)})$, which can be accomplished by two gridding steps (per coil). We will now show how to remove the two gridding interpolation steps, and since the calculation time is heavily determined by these two steps we get a major speedup.

Instead of using $\mathrm{E}^H\mathrm{E}\mathbf{I}^{(n)} = \mathrm{E}^H(\mathrm{E}\mathbf{I}^{(n)})$ we examine the components of $\mathrm{E}^H\mathrm{E}$,

$$[\mathrm{E}^H\mathrm{E}]_{\rho,\rho'} = \sum_\gamma s^*_\gamma(\mathbf{r}_\rho)\left(\sum_\kappa e^{-i\mathbf{k}_\kappa\cdot(\mathbf{r}_\rho-\mathbf{r}_{\rho'})}\right)s_\gamma(\mathbf{r}_{\rho'}) \tag{3}$$

$$= \sum_\gamma s^*_\gamma(\mathbf{r}_\rho)\mathrm{Q}(\mathbf{r}_\rho-\mathbf{r}_{\rho'})s_\gamma(\mathbf{r}_{\rho'}), \tag{4}$$

with

$$\mathrm{Q}(\mathbf{r}) = \sum_\kappa e^{-i\mathbf{k}_\kappa\cdot\mathbf{r}}. \tag{5}$$

Note the similarity between the summation in Eq. 5 and the term in parentheses in Eq. 2. They can be evaluated at the same time prior to starting the iterations using gridding. The only thing we need to take care about is that the size of Q is twice the size of $\mathbf{I}^{(n)}$, since $\mathbf{r}_\rho - \mathbf{r}_{\rho'}$ has twice the range of $\mathbf{r}_\rho$.

The elements of $\mathrm{E}^H\mathrm{E}\mathbf{I}^{(n)}$ are now evaluated as

$$[\mathrm{E}^H\mathrm{E}\mathbf{I}^{(n)}]_\rho = \sum_\gamma s^*_\gamma(\mathbf{r}_\rho)\left(\sum_{\rho'}\mathrm{Q}(\mathbf{r}_\rho-\mathbf{r}_{\rho'})s_\gamma(\mathbf{r}_{\rho'})I^{(n)}_{\rho'}\right) \tag{6}$$

$$= \sum_\gamma s^*_\gamma(\mathbf{r}_\rho)\left[\mathrm{Q} * (s_\gamma(\mathbf{r}_{\rho'})I^{(n)}_{\rho'})\right]_\rho, \tag{7}$$

where the $*$ in Eq. 7 stands for a discrete convolution [5]. This convolution can be efficiently evaluated using the FFT [6], *i.e.* $\mathrm{Q} * (s_\gamma(\mathbf{r}_{\rho'})I^{(n)}_{\rho'}) = \mathrm{IFFT}(\mathrm{FFT}(\mathrm{Q})\cdot\mathrm{FFT}(s_\gamma(\mathbf{r}_{\rho'})I^{(n)}_{\rho'}))$. Instead of storing Q, FFT(Q) is stored, which leads to $\mathrm{Q} * (s_\gamma(\mathbf{r}_{\rho'})I^{(n)}_{\rho'})$ being evaluated using only two FFT's. The size of these FFT's is equal to the size of Q, requiring $s_\gamma(\mathbf{r}_{\rho'})I^{(n)}_{\rho'}$ to be zero-filled before applying the FFT. This size is the same as used in [4], so there is no increase in calculation time of the FFT's. Therefore, the gain is that we have replaced two gridding interpolations (per coil) by a coil-wise multiplication in k-space by FFT(Q), which can be evaluated much faster.

Finally, including a diagonal preconditioning matrix D into the algorithm, as suggested in [4], *i.e.* $\mathrm{E}^H\mathrm{DE}\mathbf{I} = \mathrm{E}^H\mathrm{D}\mathbf{m}$, presents no problem and does not increase computation time. In Eq. 2 we include an extra term $\mathrm{D}_{\kappa,\kappa}$ and the definition of Q changes to

$$\mathrm{Q}(\mathbf{r}_\rho) = \sum_\kappa \mathrm{D}_{\kappa,\kappa}e^{-i\mathbf{k}_\kappa\cdot\mathbf{r}_\rho}. \tag{8}$$

Results

The proposed convolution approach was implemented in the iteration scheme described in [4]. Its performance was compared with that of the original method using explicit gridding. Single-shot spiral acquisition with radial density reduced to 40% of the Nyquist limit (R=2.5) was simulated using an array of six circular receiver coils surrounding a cylindrical phantom. All calculations were performed on a 400 MHz DEC Alpha workstation using PV Wave (Visual Numerics Inc., Houston, Texas).

The speed benefit of the new approach was assessed for typical image sizes. The results are summarized in Tab. 1, proving a substantial speed advantage of the convolution method. As expected, except for negligible numerical error, the novel implementation yielded exactly the same image progressions as did the original method.

Table 1: Comparison of CPU times per iteration.

Image size	64 × 64	128 × 128	256 × 256
Convolution	1.1 s	2.7 s	12 s
Gridding	4.5 s	17 s	68 s

Conclusions

The novel method is really fast (gain of $3-6\times$) and makes SENSE in combination with spiral even more practical. The actual speed benefit is greatest for low reduction factors, because gridding is most costly with dense k-space sampling.

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