LORAKS: Low-Rank Modeling of Local *k*-Space Neighborhoods

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Introduction: Low-rank matrix representations of MRI data have recently become popular in high-dimensional constrained image reconstruction for applications like dynamic MRI [1-3] and parallel MRI [4], motivated by theory that shows that low-rank matrices can often be accurately reconstructed from incomplete measurements [1,5]. In these applications, low-rank modeling is enabled by the relatively high-dimensionality of the data, and has been observed to outperform/enhance constrained reconstructions that solely rely on image sparsity. However, application of low-rank modeling to single-channel, single-contrast, single-timepoint MRI images is more challenging, due to the much lower-dimensionality of the reconstruction problem.

In this work, we observe that single (low-dimensional) MRI images can be mapped to much higher-dimensional low-rank matrices when the images have limited spatial support or slowly-varying phase. This enables a novel and flexible framework for constrained image reconstruction that uses low-rank matrix modeling of local *k*-space neighborhoods (LORAKS). The LORAKS approach is analogous to a single-channel version of GRAPPA [6], except that LORAKS can be used with calibrationless *k*-space trajectories, uses information from local neighborhoods on both sides of *k*-space, is developed from first-principles (which provides new perspectives on methods like GRAPPA [6] and SAKE [4]), and is flexible enough to be used with other commonly-used constraints like image sparsity. Related work that takes a similar approach was recently described in [7].

Theory and Methods: In the following, we denote the image as $\rho(\mathbf{x})$ with corresponding Nyquist-sampled *k*-space data $s[\mathbf{n}]$. LORAKS is based on the fact that multiplication in the image domain is equivalent to convolution in the Fourier domain, and that a real-valued image has conjugate symmetry in *k*-space. These facts can be used to construct three different GRAPPA-like linear dependence relationships, as follows: (R1) If an image has identically-zero regions within the FOV, then there exists at least one non-zero function $f(\mathbf{x})$ such that $\rho(\mathbf{x})f(\mathbf{x}) = 0$. This implies that there is at least one non-zero *k*-space function $\tilde{f}[\mathbf{n}]$ such that the convolution $\sum_{\mathbf{p}} s[\mathbf{n}-\mathbf{p}]\tilde{f}[\mathbf{p}] = 0$ for all possible choices of $\mathbf{n}$ [8]. This means that a rank-deficient matrix $\mathbf{C}$ can be obtained from $s[\mathbf{n}]$ by choosing the row elements of $\mathbf{C}$ to equal $s[\mathbf{n}-\mathbf{p}]$ for different choices of $\mathbf{p}$, with each row corresponding to a different choice of $\mathbf{n}$. Note that the $\mathbf{C}$ matrix is a single-channel version of the matrix used in SAKE [4]. (R2) There always exists at least one non-zero function $h(\mathbf{x})$ such that $\rho(\mathbf{x})h(\mathbf{x})$ is real-valued (for example, $h(\mathbf{x})$ can be constructed using the inverse of the phase of $\rho(\mathbf{x})$). This implies that there is at least one non-zero *k*-space function $\tilde{h}[\mathbf{n}]$ such that $\sum_{\mathbf{p}} s[\mathbf{n}-\mathbf{p}]\tilde{h}[\mathbf{p}] = \sum_{\mathbf{p}} s^*[-\mathbf{n}-\mathbf{p}]\tilde{h}^*[\mathbf{p}]$ for all possible choices of $\mathbf{n}$. Similar to the previous case, this means that we can construct a rank-deficient matrix $\mathbf{S}$ with row elements equal to $s[\mathbf{n}-\mathbf{p}]$ and $s^*[-\mathbf{n}-\mathbf{p}]$. (R3) Finally, there always exists at least one non-zero function $g(\mathbf{x})$ such that $\rho(\mathbf{x})g(\mathbf{x}) = \rho^*(\mathbf{x})$. This implies that there is at least one non-zero *k*-space function $\tilde{g}[\mathbf{n}]$ such that $s^*[-\mathbf{n}] = \sum_{p} s[\mathbf{n}-\mathbf{p}]\tilde{g}[\mathbf{n}]$ for all possible choices of $\mathbf{n}$ [9]. As before, this means that we can construct a rank-deficient matrix $\mathbf{G}$ with row elements equal to $s[\mathbf{n}-\mathbf{p}]$ and $s^*[-\mathbf{n}]$. In practice, $\mathbf{C}$, $\mathbf{G}$, and $\mathbf{S}$ are frequently low-rank rather than merely rank-deficient, since the linear dependence relationships are satisfied for multiple different choices of $\tilde{f}[\mathbf{n}]$, $\tilde{h}[\mathbf{n}]$, and $\tilde{g}[\mathbf{n}]$.

Based on these relationships, it should be possible to reconstruct an image from undersampled data by finding an estimated fully-sampled data vector $\hat{\mathbf{s}}$ that is (1) consistent with the measured data and (2) leads to low-rank $\mathbf{C}$, $\mathbf{G}$, and $\mathbf{S}$ matrices. In this work, we achieve this by finding an $\hat{\mathbf{s}}$ that minimizes $\|\mathbf{F}\hat{\mathbf{s}} - \mathbf{d}\|_{\ell_2}^2 + \lambda_C J(P_C(\hat{\mathbf{s}})) + \lambda_G J(P_G(\hat{\mathbf{s}})) + +\lambda_S J(P_S(\hat{\mathbf{s}}))$, where $\mathbf{d}$ is the measured (undersampled) *k*-space data, $\mathbf{F}$ is a matrix that models subsampling of the fully-sampled data, $P_C(\cdot)$, $P_G(\cdot)$, and $P_S(\cdot)$ are linear operators that form the $\mathbf{C}$, $\mathbf{G}$, and $\mathbf{S}$ matrices from a vector of fully-sampled *k*-space data, λ_C, λ_G, and λ_S are regularization parameters, and $J(\cdot)$ is a penalty function that encourages low-rank matrices. In this work, we choose $J(\cdot)$ to be a nonconvex function that measures the amount of error (in the Frobenius norm) that is incurred when a matrix is approximated as having rank ℓ, where ℓ is a user-chosen parameter. While this cost function is non-convex and might seem complicated, minimizers can be obtained by a simple alternating algorithm involving the iterative computation of truncated SVDs.

Results: Results are shown in Fig. 1 for a single-channel reconstruction of a retrospectively-undersampled T2-weighted spin-echo brain image of a healthy subject, using a "calibrationless" acquisition that does not sample densely at the center of *k*-space. LORAKS reconstructions are shown in comparison to total variation (TV) reconstruction, as well as ℓ_1-regularized reconstruction that assumes the image is sparse in the pixel basis (i.e., the canonical basis for the image domain). We observe that LORAKS constraints yield different (and frequently better) results than TV or ℓ_1-regularized constraints, with $\mathbf{S}$-based LORAKS being the most powerful of the three different constraints.

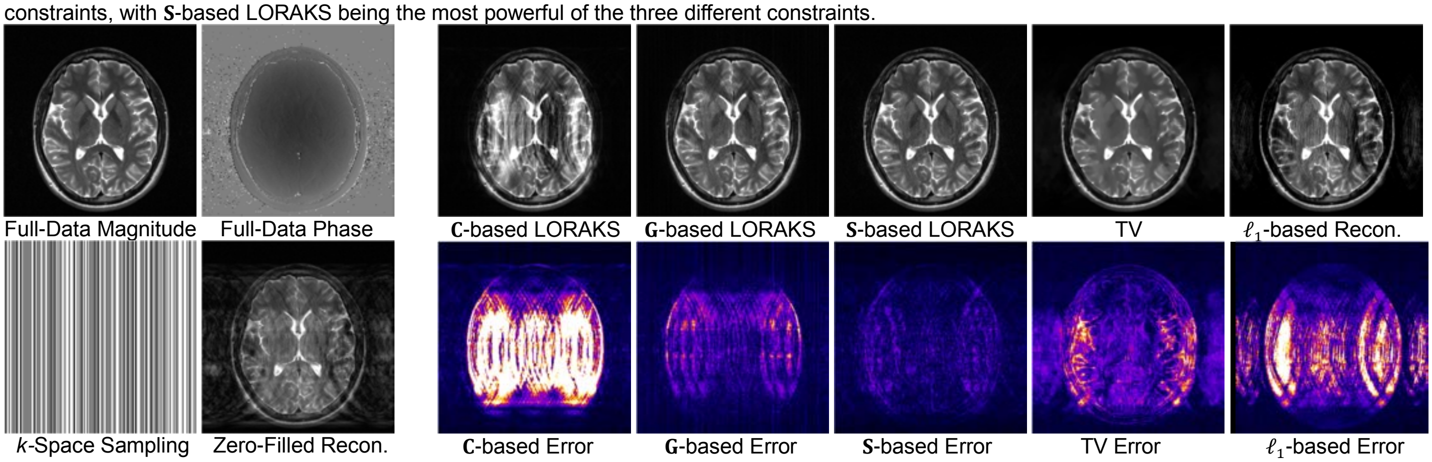

Figure 1. Comparison of different LORAKS-based reconstruction strategies with TV- and ℓ_1-regularized reconstruction from calibrationless data.

Conclusions: This work demonstrated that low-rank matrix modeling can be used in a novel way to enable calibrationless reconstruction of low-dimensional MRI images. The approach was demonstrated to have advantages over other constrained reconstruction methods in certain settings, and is easily used in concert with other forms of regularization.

References: [1] Z.-P. Liang, *IEEE ISBI*, 2007. [2] J. Haldar, *IEEE ISBI*, 2010. [3] S. Lingala, *IEEE Trans Med Imaging* 30, 2011. [4] P. Shin, *Magn Reson Med* Early View, 2013. [5] B. Recht, *SIAM Rev* 52, 2010. [6] M. Griswold, *Magn Reson Med* 47, 2002. [7] J. Haldar, *ISMRM Data Sampling*, 2013. [8] K. Cheung, *J Opt Soc Am A* 7, 1990. [9] F. Huang, *Magn Reson Med*, 2009.