INTRODUCTION: Combining k-space under-sampling with a non-linear compressed sensing (CS) algorithm is useful in magnetic resonance (MR) imaging to decrease image acquisition times while retaining resolution 1. One CS reconstruction criterion is a sparse transform domain representation. The wavelet transform can sparsify images but is not applicable when under-sampling a 4th non-k-space domain, time or frequency. In these domains a signal $R(t)$, or its discrete Fourier transform (DFT) $R(f) = DFT(R(t))$, is a sum of exponentials or Lorentzians respectively.

PURPOSE: We propose the use of the Gardner transform 2–3 (GT) as a sparsification transform for sums of exponentials, Lorentzians, Gaussians, or sinc functions. We present GT theory together with simulation results illustrating challenges arising in the practical application of the GT.

THEORY: Analysing a multi-component exponential decay, $R(t) = \sum A_i \exp(-t/\text{MTT}_i)$, using the GT requires determining the solution of the deconvolution equation, $GG(f) = \text{DFT}[R(t)] = \text{DFT}[R(t)] / \text{DFT}[\Delta(x)]$, where $GG(f)$ is obtained by the substitution of $t = \exp(-x)$ into $R(t) = \exp\{\text{MTT}_i\} \times \exp(-x)$, with the Gardner kernel, $\Delta(x) = t \exp(-t)$. $GG(x)$ is a sum of delta functions at $x_i = -\text{MTT}_i / \text{MTT}$, with magnitudes proportional to $A_i / \text{MTT}_i$ for the signal $R(t) = \sum A_i \exp(-t/\text{MTT}_i)$). $GG(x)$ can also be generated from sums of Lorentzians, Gaussian, or sinc functions.

METHODS: Smith et al. 1 showed key GT practical issues were: 1) Gardner domain signal $GG(x)$ generation from either the (exponential) time domain signal, or the (Lorentzian) frequency domain signal, and 2) generating the sparsified signal $GG(x)$ by deconvolving $RR(t)$ by the Gardner kernel $\Delta(x)$. Stabilizing the deconvolution requires removing high frequency noise components in $GG(f)$. This filtering operation also removes high frequency signal components and widens the peaks in $GG(x)$ reducing sparseness. In this investigation we focus on techniques to recover the sparseness lost by filtering by recovering the under-sampled, in this case truncated, $GG(x)$ function for an N point DFT since $GG(x)$ is a sum of exponentials of Lorentzians, Gaussians, or sinc functions. We present GT theory together with simulation results illustrating challenges arising in the practical application of the GT.

RESULTS: Figure 1A is $GT[R(t) = E_{GM} + E_{WM}]$ when all $GG(f)$ components are known. The reason behind the peak value and shape differences arise from specific characteristics of the GT frequency components of $E_{GM}$ and $E_{WM}$, figure 1-GM and 1-WM respectively. $GT[E_{GM}]$ is a basis function for an N point DFT since $MTT_{GM} = 4.055\times \exp(28x)$. However, $MTT_{WM} = 4.8\times \exp(31.4x)$ so that $GT[E_{WM}(f)]$ is not a basis function and has discontinuities at the frequency domain boundaries. $\Delta(f) = N/2$. SparseMRI CS reconstructions attempts to minimize $\|GG(f)\|_{1,1}\text{-norm}$ widening the $E_{WM}$ peak rather than sharpening it. Note the re-introduced $GG(f)$ continuity 5. Application of GT-CS on the truncated $GG(f)$ leads to significant intensity and GT resolution loss, figure 2A, and fails to recover missing high frequency components, figures 2-GM and 2-WM. This again can be explained smaller $L_1$-norms leading to short and wide, rather than tall and narrow, peaks. Preconditioning the missing $GG(f)$ using values obtained by modeling the central $GG(f)$ does not lead to an improved $GG(x)$ estimate unless a few random $GG(f)$ values are assumed known (red-spots in figures 3-GM and 3-WM). Figure 3A is almost exactly recovered to the result in figure 1A despite using half the frequency information. Note the lack of k-space continuity 5 in $E_{WM}(f)$ again leads to a wider $GG(x)$ peak in figure 3A.

CONCLUSION: We have identified the GT as a sparse representation for several non-k-space signals, e.g. multi-exponentials, and provided preliminary simulations results. Combining GT-CS with k-space extrapolation techniques 6,7 are being investigated.


Column A: $GG(x)$ CS reconstructions for: 1) 100% sampling, 2) center 30% used, 3) center 30% used and 10% random sampling on each side. Column GM and column WM: Frequency magnitude response (black), real component of $GG(f)$ (blue), and CS data consistency constraint (red) for $E_{GM}$ and $E_{WM}$ respectively.