Compressed Sensing and HYPR  
Julia Velikina, PhD  
Department of Medical Physics, University of Wisconsin – Madison

Introduction

The desire to achieve high spatial and/or temporal resolution in MRI coupled with limited scan time has led to the necessity to reconstruct images from incomplete datasets. Mathematically this amounts to solving an underdetermined system of equations, that is, the number of unknowns (image pixel values) is larger than the number of equations (acquired measurements). Underdetermined systems of equations generally have an infinite number of possible solutions. A standard approach to isolate a single feasible solution is to incorporate additional prior information about the problem in order to regularize reconstruction and to account for unsampled data points. A number of methods to accelerate MR imaging have been proposed, including parallel imaging [1-3], UNFOLD [4], k-t BLAST/SENSE [5], and, more recently, compressed sensing [6-8] and HYPR [9, 10]. We will discuss several different ways to regularize image reconstruction from incomplete data, using both theoretical assumptions and image-specific constraints.

Regularized Image Reconstruction

In MRI, a digital image needs to be estimated from a discrete set of measurements, each sample representing an integral of the image modulated by encoding function composed of magnetic field gradient and coil sensitivity modulations. In the matrix form, the signal vector is modeled by

\[ s = Ef + \varepsilon \]  \hspace{1cm} [1]

where \( f \) is the column vector of length \( M \) corresponding to image to be estimated, \( E \) is the encoding matrix that includes Fourier terms and coil sensitivities, and \( \varepsilon \) corresponds to the complex noise in the measurements. If row rank of \( E \) (the number of linearly independent rows) is less than \( M \), the number of unknowns, then the problem of estimation of \( f \) from the set of measurements is underdetermined and Eq. [1] has an infinite number of solutions. Otherwise, when \( \text{rank} \ E \geq M \), an SNR-optimized estimation of image in the presence of independent noise can be accomplished in the least squares sense:

\[ f = \arg \min_f \| Ef - s \|_2^2, \]

where \( \| \cdot \|_2 \) is \( \ell_2 \) norm. However, even in this case, the system of equations [1] may be poorly conditioned, for example, when using high undersampling factors in multicoil parallel MRI reconstruction. In both cases, estimation of the underlying signal \( f \) can be improved by constraining it by prior information about \( f \). Usually, this is done by including an additional regularization term into the object function:

\[ f = \arg \min_f \left( \| Ef - s \|_2^2 + \lambda R(f) \right). \]
Design of the penalty functional $\mathcal{R}(f)$ is governed by available knowledge or model assumptions about the image $f$ (prior information) and often by the ease of implementation/computational complexity. Parameter $\lambda$ provides balance between the data fidelity (first term) and prior information assumptions (second term).

Typically, penalty functionals are of the form

$$\mathcal{R}(f) = \| \Phi f \|_p^p,$$

where

$$\| x \|_p = \left( \sum_k |x|^p \right)^{1/p}.$$

and $\Phi$ is an appropriately chosen transform, e.g. identity transform, discrete derivative, or wavelet transform.

**Tikhonov Regularization.** One of the most common ways of regularization is Tikhonov regularization \cite{11,12} corresponding to $p = 2$:

$$f = \arg \min (\| E f - s \|_2^2 + \lambda \| f \|_2^2).$$

Such formulation is dictated by *a priori* assumptions that the measurements are contaminated by white Gaussian noise and is aimed at optimizing the noise in the solution. However, Tikhonov regularization may not be sufficient to eliminate undersampling artifacts.

**Compressed Sensing.** Recently, a novel mathematical theory has been developed \cite{6} that states that sparse images (i.e., images with a relatively small number of pixels containing relevant information) can be accurately reconstructed from undersampled datasets, provided the encoding matrix $E$ satisfies certain conditions. Ideally, the sparsity of an image is measured by its $l_0$ norm that counts the number of non-zero pixels (note that despite the name $l_0$ norm is technically not a norm as it does not satisfy all the necessary axioms). If we know in advance that the underlying image is expected to be sparse (as is the case, for example, in MR angiography), then the image may be obtained as

$$f = \arg \min (\| E f - s \|_2^2 + \lambda \| f \|_0).$$

The problem with this formulation is that while it allows for an accurate reconstruction of sparse images, its practical implementation is complicated since the direct approach has combinatorial complexity. However, the compressed sensing theory proves that, under certain conditions, the solution of $l_0$ minimization problem is equivalent to the solution of $l_1$ minimization problem, i.e. we can solve the following problem:

$$f = \arg \min (\| E f - s \|_2^2 + \lambda \| f \|_1).$$

There are a number of computationally efficient ways to implement $l_1$ minimization in practice, which made compressed sensing ideas attractive to accelerated MR imaging \cite{7,8}.
In compressed sensing, admissible acceleration factors are analytically related to the sparsity level of the underlying signal. Higher level of undersampling leads to artifacts in the reconstructed images. This often poses a problem in rapid imaging, since even intrinsically sparse angiographic images may not possess the level of sparsity necessary to support the high acceleration factors desirable in some applications. However, image sparsity can be enhanced either by an application of a sparsifying transform such as an image gradient or a wavelet transform, or by subtracting a prior image estimate [13], or by both. Moreover, several regularizing terms may be used to provide a better reconstruction. Therefore, in its most general formulation compressed sensing solves the following minimization problem:

\[ f = \arg \min \left( \| \mathbf{E} f - s \|_2^2 + \sum \lambda_k \| \Phi_k (f - f_{0,k}) \|_1 \right). \]

Here, \( \Phi_k \) are sparsifying transforms and \( f_{0,k} \) are corresponding prior image estimates that may be obtained in a number of ways, for example, from a prior scan, or from more densely sampled low frequencies, or from temporally averaging a time series.

Images in Fig. 1 compare the effects of different ways to regularize reconstruction and provide an illustration to the fact that \( l_2 \) norm is optimal from the point of view of noise properties but does not eliminate residual artifacts; \( l_1 \) norm helps restore image sharpness but is not noise optimal; while \( l_1 \) norm of the image gradient provides a tradeoff between these two cases.

We will discuss both theoretical requirements of compressed sensing and some aspects of its practical implementation.

Figure 1. Reconstruction of an image from a radially undersampled dataset (acceleration factor 4) using \( l_2 \) regularization (left), \( l_1 \) regularization of the image itself (center), and \( l_1 \) regularization of the image gradient (right).
The HYPR method belongs to another family of constrained reconstruction algorithms, which use a multiplicative constraint by a prior image. HYPR reconstruction is usually applied to serial imaging, such as time-resolved imaging or diffusion tensor imaging. A HYPR image is obtained as

\[ H = W \cdot C \]

where \( C \) is the prior image estimate and \( W \) is a weighting image. The prior image estimate, \( C \), also called the composite image, is usually obtained from averaging all or a subset of the data collected during the exam. The quality of the composite image largely determines spatial resolution and SNR of the individual HYPR frames. The main distinction between different algorithms in the HYPR family lies in the way the weighting images are formed. The original HYPR algorithm [9] and its modification [14] use unfiltered backprojection, and therefore are tailored specifically to radial acquisition. The subsequently developed HYPR LR algorithm [10] relies on k-space filtering to form the weighting images and can be applicable to any sampling trajectory. Another advantage of HYPR LR is that it reduces signal cross-talk between spatially adjacent objects with different time courses, such as, for example, an artery and a vein. This property allows for the use of composite images collected over a longer period of time and, thus, having higher SNR, which is then transferred to individual HYPR frames. Another possibility is to acquire a composite image in a separate scan, as was done in the HYPR Flow technique [15]. Images in Fig. 2 illustrate contrast arrival in an AVM patient using the HYPR Flow technique.

Both HYPR and HYPR LR algorithms are approximate image reconstruction techniques. We will discuss the dependence of the reconstruction error and performance of the algorithms on image sparsity and spatio-temporal correlation of the images in the series. We will also discuss several iterative HYPR techniques [16-18] that were developed with the aim of improving the accuracy of the reconstruction.

References


