Global Minimum Peak RF Design for Large Time-Bandwidth Saturation Pulse

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\textbf{Introduction} Saturation pulses $r(t)$ are essential to many imaging applications [1-3]. Criteria for desirable saturation profile $|M_z|$ are that passband and sharp profile with minimum peak $r_f(t)$ amplitude. Design parameters for RF pulses include passband $\delta_1$ and stopband $\delta_2$ ripple tolerances and time-bandwidth product $tb$ [1]. The well-known Shinmar-Le Roux (SLR) RF pulse design technique is a transform that relates magnetization profile to two polynomials $A_N$ and $B_N$ [4-6]. $B_N$ has been obtained, in the past, by traditional digital filter design techniques using the parameters above as input. A conventional approach (for minimum-peak $r_f(t)$) is to design a maximum-phase polynomial $B_N$, factor $B_N$ to obtain its roots, then combinatorially search by root inversion [7] over all possible phase patterns. But this conventional method is limited to $tb \approx 18$ before number of combinations becomes prohibitive. For $tb$ well in excess of that, we propose a novel Optimization technique that determines a $B_N$ yielding the global minimum peak $r_f(t)$ amplitude.

\textbf{Methods} For saturation pulses, the RF pulse $r_f(t)$ and polynomial $B_N$ are essentially the same. We first use Optimization to generate a minimum peak $B_N$, then the saturation profile $|M_z|$ and RF pulse $r_f(t)$ are found via SLR transform as in the conventional method. We want to find a minimum peak $B_N$ whose frequency response $R(\omega)$ satisfies design specifications, as formulated in Eq. 1. But this problem statement is nonconvex (i.e. solution not necessarily globally optimal). So instead, define an autocorrelation matrix of $B_N$ as $G \triangleq B_N B_N^T \in \mathbb{C}^{N \times N}$, where $G$ is positive semidefinite with rank 1. Summing along each of $2N-1$ subdiagonals produces entries of the autocorrelation function $r$ of $B_N$, where $r \triangleq r_n + r_{-n} \in \mathbb{C}^{N}$. In particular, the main diagonal of $G$ holds squared absolute entries of $B_N$ (Fig. 1). Minimizing $\|B_N\|_\infty$ is therefore equivalent to minimizing $\|\text{diag}(G)\|_\infty$. Define $I_0 \equiv 1$ and define $I_n$ as a zero matrix having vector 1 along the $n^{th}$ superdiagonal when $n$ is positive or 1 along the $n^{th}$ subdiagonal when $n$ is negative. By spectral factorization [8], $R(\omega) = |H(\omega)|^2$, an equivalent problem is expressed in Eq. 2.

\begin{equation}
\text{minimize } \|B_N\|_\infty \\
\text{subject to } 1 - \delta_1 \leq |H(\omega)| = 1 + \delta_1, \quad \omega \in [0, \omega_p] \\
|H(\omega)| \leq \delta_2, \quad \omega \in [\omega_s, \pi]
\end{equation}

Eq. 1

\begin{equation}
\text{minimize}_{G, r} \|\text{diag}(G)\|_\infty \\
\text{subject to } R(\omega) = r_n(0) + 2 \sum_{n=1}^{N-1} r(n) \cos(n) \\
(1 - \delta_1)^2 \leq R(\omega) \leq (1 + \delta_1)^2, \quad \omega \in [-\omega_p, \omega_p] \\
R(\omega) \geq 0, \quad \omega \in [-\pi, \pi] \\
r(n) = \text{trace}(I_n^T G), \quad n = 0 \ldots N-1 \\
G \succeq 0 \\
\text{rank}(G) = 1
\end{equation}

Eq. 2

Excepting the rank constraint, this problem is convex; this means, it is numerically solvable for its global minimum by prevalent and readily available software programs. The rank constraint is met via method of convex iteration. Polynomial $B_N$ is taken from the first column of $G$. Having $B_N$, the RF waveform $r_f(t)$ is found by SLR transform. We use our optimization to design a saturation pulse with $tb=14$ and then compare peak RF amplitude with those from a maximum-phase design with and without combinatorial search of all root inversion. We show another example of a large time-bandwidth design ($tb=32$) for which it is not practical to obtain minimum peak RF amplitude by root inversion.

\textbf{Results} Figure 2 (left) shows RF waveforms $r_f(t)$ from the optimization (red), and maximum-phase design with (blue) and without (black) root inversion. The design parameters are: $tb=14$ (pulse duration $= 4ms$), $\delta_1=2\%$, $\delta_2=0.1\%$. Figure 2 (right) shows their corresponding $|M_z|$, where $|M_z| = 1 - 2|\beta|^2$ and $\beta$ is the Cayley-Klein parameter obtained from SLR transform. All three designs meet specifications, while the optimization technique has lowest peak RF amplitude.

\textbf{Discussion} We presented a method to design an RF saturation pulse with globally minimum peak RF power without the need for exhaustive root-inversion search. For saturation RF pulses $r_f(t) \approx B_N$, minimum peak $r_f(t)$ can be obtained by Convex Optimization for which the objective is to find a $B_N$ having the lowest possible peak amplitude while simultaneously satisfying the given filter design parameters. Instead of working directly with impulse response, our technique employs its autocorrelation and imposes a rank constraint on the autocorrelation matrix $G$. The main diagonal of $G$ is $[B_N(0)B_N(0)^H, B_N(1)B_N(1)^H, \ldots]^T$. Minimizing $\|B_N\|_\infty$ is therefore equivalent to minimizing $\|\text{diag}(G)\|_\infty$. Once the program has achieved global minimum, $B_N$ can be extracted from the first column of $G$. Columns of autocorrelation matrix $G$ are simply copies of $B_N$ having different scale factors; consequently, $\text{rank}(G) = 1$. But a rank constraint is not convex. We overcome this by using the method of convex iteration. The number of iterations varies but total computation time is usually on the order of minutes in Matlab. Since RF power is directly related to SAR safety limits, this method is most relevant in high field applications and large time-bandwidth-product designs. Possible applications include TOF (selective arterial or venous imaging), outer volume suppression pulses, spectral saturation/suppression.