Comparative Evaluation of $l_1$ vs $l_p$ Minimization Techniques for Compressed Sensing MRI

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Introduction
Compressed sensing (CS) has been shown to provide accurate reconstructions from highly undersampled data for certain types of MR acquisitions [1, 2]. This offers the promise of faster MR acquisitions, and further speed gains are possible when CS is used in conjunction with parallel acquisition schemes such as SENSE [3]. Several approaches have been recently proposed to reconstruct images from even fewer measurements than those required by standard $l_1$-norm compressed sensing [2,4,5]. The purpose of this study was to test and compare standard $l_1$-norm CS with two such approaches based on the $l_p$ quasi-norm [2,4,6] across different sampling pattern densities and parameterizations.

Theory
CS provides a method to reconstruct signals that are sampled well below the Nyquist limit, if the signal has a sparse representation under some transform $\Psi$, by selecting the solution with the sparsest representation in $\Psi$ which still matches the limited observation set $f$. Such a search consists of solving the following equality-constrained $l_p$-minimization problem:

$$\min_{\Phi\in \Phi^-} \|\Phi u\|_p \quad \text{s.t.} \quad \Phi u = f$$

Direct addressing of the $l_p$-minimization problem requires a combinatorial search of potential solutions which is intractable for most practical applications. In CS the $l_p$-quasi-norm is replaced by the convex $l_1$-norm, and (1) becomes a tractable minimization at the cost of requiring a modest degree of oversampling over the theoretical minimum required by the $l_p$ case. A natural question that arises is whether it is possible to find an alternative prior one can use to get closer to the $l_p$-quasi-norm bound. Chartrand [4] proposed the use of the $l_p$-norm, where $0<p<1$. The corresponding $l_p$-norm recovery problem becomes:

$$\min_{\Phi\in \Phi^-} \|\Phi u\|_p \quad \text{s.t.} \quad \Phi u = f$$

where $\|\Phi u\|_p = \sum_{i=1}^{N} |\Phi u_i|^p$ and $\varepsilon$ is a smoothing parameter. In [7] Chartrand and Yin proposed a continuation approach on $\varepsilon$. More recently, Trzasko and Manduca [2] presented a generalization of the CS paradigm based on a homotopic approximation of the $l_0$ quasi-norm, $H_{\ell_0}$, which allows the use of different functionals. Here we consider $H_{\ell_0}$ minimization using the same $l_p$ quasi-norm, with fixed $\varepsilon$ but effectively doing continuation on $\varepsilon$. Since both of these approaches involve non-convex $l_p$ minimization, there is no theoretical guarantee of convergence to the optimal solution as there is in the $l_1$ case.

Methods
We implemented the $l_1$, $l_p$, and $H_{\ell_0}$ algorithms using the spatial gradient as the sparsifying transform and tested them using a 128x128 Shepp-Logan phantom with 11 different undersampling rates $\text{USR} = 0.75, 0.80, 0.85, 0.88, 0.90, 0.92, 0.93, 0.94, 0.95, 0.96, 0.97$ and 0.98. Noting the use of variable-density k-space sampling strategies in [1] and [2], we also tested 10 different Gaussian k-space sampling parameterizations (standard deviation = N/R, for R = 1:10). For the $l_0$ algorithm we studied 10 parameterizations from $0.1c<1.0$. The $H_{\ell_0}$ algorithm was tested with $\varepsilon$ between $10^{-10}$ and $10^{-5}$. We repeated each experiment 10 times with different random sampling patterns, performing 22,000 simulations in all. Reconstructions were assessed by calculating the root minimum, mean and maximum squared errors (RMNRMSE, RMSE, RMXSE) per pixel.

Conclusions
The simulations confirmed that it is possible to substantially reduce the number of samples required for exact reconstruction when using the $l_p$-norm for values $p<1$. Standard $l_1$-norm CS broke down at $\text{USR}=0.93-0.94$, whereas both the $l_p$ and $H_{\ell_0}$ approaches could achieve accurate reconstructions for some $R$ at $\text{USR}=0.96-0.97$. Despite the lack of theoretical guarantees of convergence, both $l_p$-norm approaches always outperformed standard $l_1$-norm CS. The simulation results suggest that the $H_{\ell_0}$-approach works well under a broader range of sampling densities $R$, but it is sensitive to the choice of $\varepsilon$, while the $l_p$-norm is fairly insensitive to the choice of $p$, but is somewhat sensitive to the choice of $R$. At high USRs $-0.95$, our simulations agree with previous observations that the $l_p$-approach does not provide better results as $p$ goes below 0.5 [7], although this was not true at higher sampling rates below $\text{USR}=0.92$. When studying the location of the pixel errors at parameterizations at which each method was beginning to fail, we consistently observed that for both non-convex $l_p$ minimizations the error was concentrated in small, low contrast objects, while the $l_1$-norm CS error was spread more widely across the image, and preferentially at high contrast borders and regions (see Figure 3). Further study investigating the properties of these approaches is required.

References