

# Gridding Reconstruction Using Optimal, Shift-Variant Interpolating Kernels

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**Introduction:** Gridding reconstruction is a method to obtain data on a rectangular grid from a set of non-uniformly sampled measurements [1,2]. This method is widely used in MR image reconstruction when the  $k$ -space sampling density is not uniform. Gridding reconstruction is well known for being robust and computationally fast. Although this method is intuitively appealing, there is no clear understanding of its optimality. On the other hand, least-square reconstruction [3,4] is another method which is optimal in the sense that it minimizes the energy of the reconstruction error. However, this method is computationally intensive and, in some cases, sensitive to measurement noise.

We show that gridding reconstruction can be considered as an approximation to the least-square reconstruction method. We also present a method to do this approximation optimally by minimizing the norm of a proper error matrix. With this method, one can calculate the optimal gridding parameters like the density compensation factors, the interpolating kernels and the de-apodization function. This method can be easily extended for generalized gridding using shift-variant interpolating kernels.

**Method:** The  $k$ -space measurements vector  $s$  is linearly related to the vector of Cartesian  $k$ -space grid points  $m$  as  $s = Hm$ ; where each column of the matrix  $H$  is the set of trajectory samples of a *sinc* kernel centered at each grid point. In least-square reconstruction, this set of linear equations is solved to obtain the grid samples as

$$\hat{m} = H^* R s \quad (1)$$

where  $R$  is obtained from the singular value decomposition of  $H$  and  $H^*$  denotes the conjugate transpose of  $H$ . In gridding reconstruction, on the other hand, the acquired data is, first, normalized to compensate for the variable sampling density of the measurement points. Then, the Cartesian samples are calculated from the normalized measurements using an interpolating kernel. With an ideal *sinc* kernel, the gridding operations can be expressed in the following matrix form:

$$\hat{m} = H^* P s \quad (2)$$

where  $P$  is a diagonal matrix with density compensation factors as the diagonal elements. Comparing these two equations, it is clear that gridding reconstruction can be derived from the least-square solution when the matrix  $R$  is properly approximated with a diagonal matrix  $P$ . The approximation error can be minimized by choosing  $P$  such that the norm of error matrix  $H^*(R - P)$  is minimized. It can be shown that the optimal elements of  $P$  are, in fact, closely related to the conventional density compensation factors.

Because of the long side lobes, the *sinc* function is not a desirable interpolating kernel and a narrower kernel, like the Kaiser-Bessel function, is most often used in practice. A narrow kernel, however, results in intensity roll-off in the reconstructed image which has to be compensated for with proper de-apodization. In matrix form, the narrow kernel corresponds to a band matrix that is to replace  $H$  in eq. [2], and the de-apodization operation corresponds to extra multiplications involving a diagonal matrix. The joint optimization of the new interpolating and de-apodization matrices is a nonlinear problem. Nevertheless, the same approach used to obtain the optimal density compensation matrix, can also be used iteratively to find the best interpolating and de-apodization matrices. In each iteration, the optimal interpolating (de-apodization) matrix is calculated given the best de-apodization (interpolating) matrix found in the previous step. Since the error does not increase in each step, this method is guaranteed to converge to

a local minimum. Note that, with this method no particular structure is imposed on the interpolating matrix (besides being a band matrix). Therefore, the interpolating kernel in this method, unlike the conventional gridding reconstruction, can be shift-variant.

**Results:** We study a case of 1D acquisition with non-uniform sampling of the  $k$ -space. We consider an exponential profile for the velocity of  $k$ -space coverage as shown in fig. 1. This profile results in a highly over-sampled region around the  $k$ -space origin. The optimal interpolating kernels of width  $3/\text{FOV}$  are calculated and compared with a Kaiser-Bessel kernel of the same width with the shaping parameter  $\beta = 2.4$ . Figure 2 shows the optimal kernel at the center of  $k$ -space and also the optimal de-apodization function. The signal to average reconstruction error ratio (SER) in the point spread function for the Kaiser-Bessel kernel is about 15.9 dB. The SER increases to 19.6 dB using the optimal kernels. This number for the ideal *sinc* kernel is 24.5 dB.

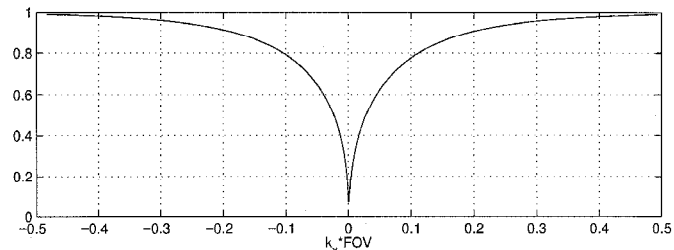


Figure 1: The normalized velocity of the  $k$ -space coverage.

**Conclusions:** We have established the relationship between the gridding and least-square reconstruction methods. We have presented a method to optimally derive gridding parameters from their counterparts in the least-square solution. Our studies have indicated that the conventional gridding parameters are close to optimal. We have shown that our method can generalize the gridding method by using shift-variant kernels resulting in about 4 dB reduction in the reconstruction error.

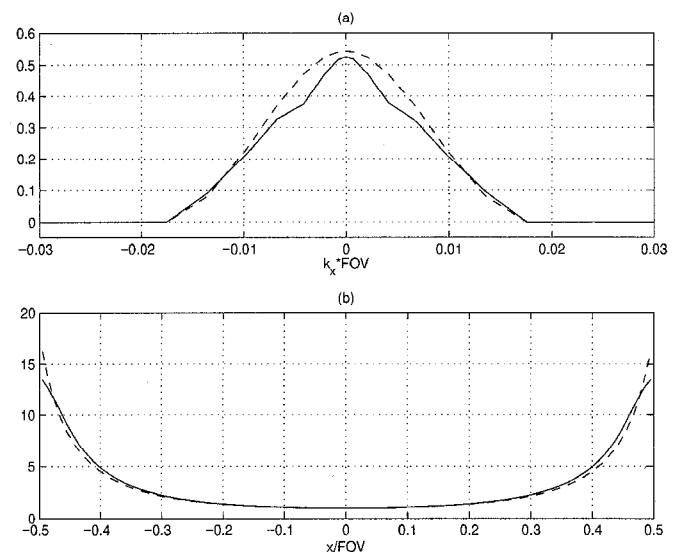


Figure 2: a) The optimal (solid) and the Kaiser-Bessel (dashed) interpolating kernels. b) The corresponding de-apodization functions.

## References:

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- [4] ROSENFELD, D., *et al.*, *MRM*, 40:14-23 (1998).